AoPS Mock Physics Competition

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2020 $F = ma$ Exam Solutions

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU START YOUR TIMER

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Test under olympiad conditions. Meaning that you must complete the test in 75 minutes in one sitting.
- This test contains 25 multiple choice questions. **Your answer to each question must be marked on the Google Forms answer sheet that accompanies the test.** Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- Correct answers will be awarded one point; incorrect answers and leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after January 28, 2020.**

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

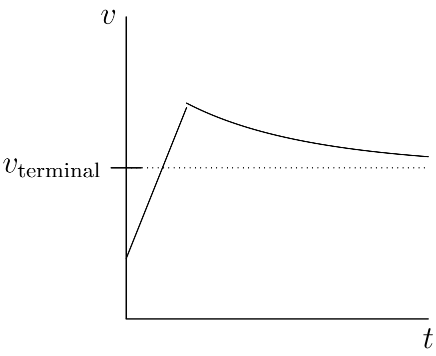
Ashmit Dutta, Aryansh Shrivastava, Viraj Jayam, QiLin Xue

Problem	**Answer**	**Topic**	**Difficulty**
1	(E)	Kinematics	3
2	(B)	Rotation	1
3	(B)	Error Analysis	2
4	(B)	Friction	4
5	(D)	Rotation, Moment of Inertia	5
6	(E)	Fluid Dynamics, Pressure	5
7	(C)	Energy, Conceptual	4
8	(B)	Springs, Conceptual	8
9	(D)	Dimensional Analysis	6
10	(B)	Fluids, Rotation	7
11	(C)	Moment of Inertia, Energy	7.5
12	(C)	Gravitation	5
13	(D)	Error Analysis	7
14	(D)	Momentum, SHM	7
15	(C)	Non-inertial Reference Frame	6.5
16	(B)	Moment of Inertia	9
17	(D)	Normal Force, Forces	7
18	(B)	Young's Modulus	8
19	(E)	Oscillations	8.5
20	(E)	Impulse	8
21	(A)	Impulse, Momentum, Gravitation	9
22	(C)	Infinite Geometric Series, Force Analysis	9.5
23	(B)	Fluids, Torque, Inequalities	9.5
24	(A)	Error Analysis	7.5
25	(B)	Fluids, Oscillators	8

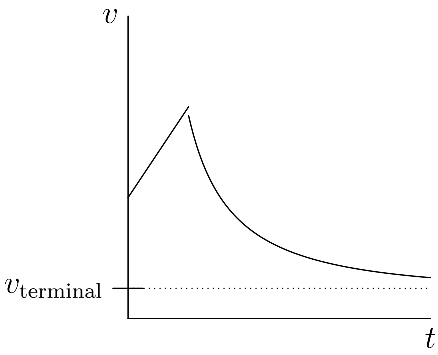
1. The 2020 US Physics Team wanted to avoid the hassle of airport baggage, so they built a rocket to take their belongings to Lithuania. The rocket is able to reach space, though they realized they forgot to attach parachutes just as it launched from Washington D.C. Let $t = 0$ be the time when the rocket is beginning to descend.

   Which of the following graphs best represents the speed of the rocket? You may assume the initial speed is near orbital speeds.

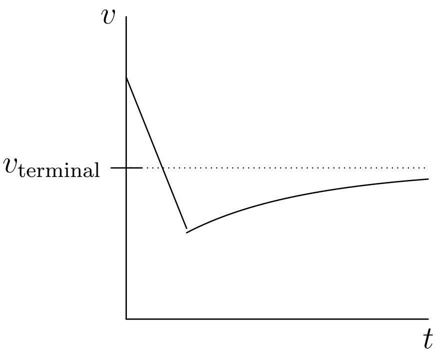
   (A)



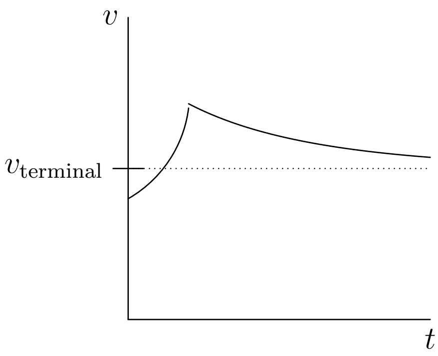
   (B)



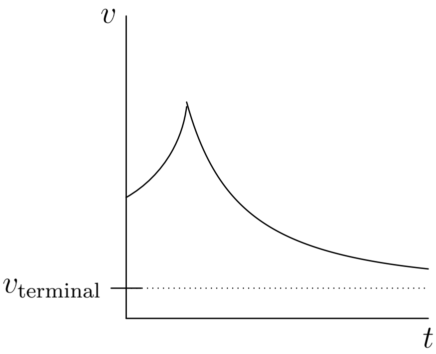
   (C)



   (D)



   (E) $\leftarrow$ Correct



Solution: As the spacecraft gets closer to Earth, the gravitational force will increase so it will be experiencing a nonuniform acceleration exponentially increasing the speed. Therefore the speed during the first part will be concave up. This leads us with D and E.

As it enters the atmosphere, the spacecraft will reach a terminal velocity. This terminal velocity will be much smaller than orbital velocity so the correct answer would be (E).

2. Felix takes a ride in a Ferris wheel. The carriage records his "weight" in kg once he sits inside. Given that the Ferris wheel has a constant radius r, with what angular velocity should the Ferris wheel move such that Felix's weight at the top of the Ferris wheel is recorded as 0 kg on the scale?

   (A) gr

   (B) $\sqrt{\frac{g}{r}}$ ← Correct

   (C) $\frac{2g}{r}$

   (D) $\frac{1}{2}gr^2$

   (E) It is impossible for Felix to have a weight of 0 kg, as he exerts a constant force of mg down on the scale wherever he goes.

Solution: The acceleration at the top of the Ferris wheel is $\omega^2 r$ which is solely provided by the gravitational acceleration when the person feels weightless. Equating:

$$\omega^2 r = g \equiv \omega = \boxed{\sqrt{\frac{g}{r}}}$$

3. It is 2050, and a robot is attempting to measure the total length of a classroom. To do so, it picks up a meter stick and starts measuring from the first end. Due to the length of the meter stick, it must make a mark after it has measured about a meter, and then move the end of the meter stick to that mark to make the next measurement. Assume that the robot's measurement is exactly perfect and that the meter stick is poorly calibrated to an order that it suspects of 5%. What is the uncertainty in this measurement if the robot finds the classroom to be about 14.54 m wide?

   (A) 0.27 m

   (B) 0.72 m ← Correct

   (C) 0.83 m

   (D) 1.03 m

   (E) 1.34 m

Solution: The key to this problem is to note that the errors in the weights are not independent. If the first measurement is too long, the second, third, etc. samples' length are also too long, and if the first sample's measurement is too short, the remaining samples' lengths are also too short. Therefore, it is not appropriate to add error in quadrature like we would if the errors were random and independent.

The approach is now a very simple one. To find the maximum error in the experiment we assume that the meter stick is poorly calibrated to the extent of 5%. Then we simply multiply

$$0.05 \cdot 14.54 \text{ m} = \boxed{0.72 \text{ m}}$$

4. Jan throws a bowling ball with radius 0.6 m so that at some point in time it moves 2 m/s leftward while rotating 4 rad/s counterclockwise about its center of mass. At this point in time, what is the direction and type of friction acting on the ball at its point of contact with the floor?

   (A) Leftward static friction

(B) Leftward kinetic friction $\leftarrow$ Correct

(C) Rightward static friction

(D) Rightward kinetic friction

(E) Friction does not act

Solution: The speed of the bottom of the ball relative to the center of the ball is

$$v = r\omega = 2.4 \text{ m/s [rightwards]}$$

However, the center of the ball is only travelling at a speed of 2 m/s [leftwards]. Thus in the frame of the ground, the bottom of the ball is actually travelling rightwards and thus the kinetic friction it experiences is to the left.

5. A toy electric top is comprised of two parts: a plastic disk of mass m_1 and radius r, and a thin metal rod of mass m_2 and length ℓ attached to the bottom side of the disk perpendicular at its center. A small generator encased inside the top allows it to rotate with an average power P about the rod from time $t = 0$ to $t = t_0$. The top does not precess. At time $t = t_0$, the angular momentum of the top about the rod is

(A) $\sqrt{Pt_0(m_1r^2 + m_2\ell^2)}$.

(B) $\sqrt{Pt_0\left(m_1r^2 + \frac{2}{3}m_2\ell^2\right)}$.

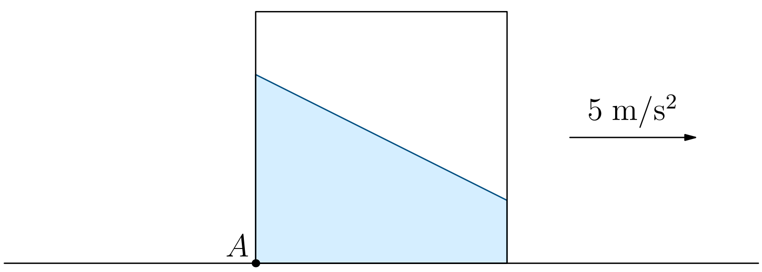
(C) $\sqrt{Pt_0\left(m_1r^2 + \frac{1}{6}m_2\ell^2\right)}$.

(D) $r\sqrt{m_1Pt_0}$. $\leftarrow$ Correct

(E) $\ell\sqrt{m_2Pt_0}$.

Solution: Average power is work over a period of time. In this case, work is done to increase the top's rotational kinetic energy from 0 to $\frac{1}{2}I\omega^2$. Hence, $\frac{1}{2}I\omega^2 = Pt_0 \implies \omega = \sqrt{\frac{2Pt_0}{I}}$. Angular momentum is $L = I\omega = \sqrt{2IPt_0}$. Now, we must determine I. The rod contributes no moment of inertia, so the only moment of inertia comes from the disk: $\frac{1}{2}m_1r^2$. In all, $L = \boxed{r\sqrt{m_1Pt_0}}$.

6. An aquarium in the shape of a cube with edges of length 2 m is half-filled with water of density 1000 kg/m^3 and sits on a frictionless surface. What is the pressure at the bottom left corner of the aquarium (denoted by point A) if the aquarium moves in the positive horizontal direction with acceleration 5 m/s^2.



(A) 1400 Pa

(B) 6100 Pa

(C) 9900 Pa

(D) 12000 Pa

(E) 15000 Pa $\leftarrow$ Correct

Solution: The total mass of the cube is

$$\frac{2^3}{2} \cdot 1000 = 4000 \text{ kg}$$

Thus, the force on the block is $4000 \cdot 5 = 20000$ N. The pressure on the bottom of the cube is

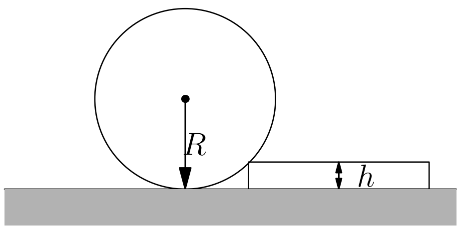
$$1000 \cdot 10 \cdot 1 = 10000 \text{ N/m}^2$$

By Pascal's Principle this pressure is the same everywhere. Let the height of the water above A be h, therefore the area of water on the left side is $2h$, and the area of water on the right side is $4 - 2h$. The net force is then

$$(4h - 4)(10000) \text{ N}.$$

This has to cancel out the 20000 N of force on the block and so $4h - 4 = 2 \implies h = 1.5$. Therefore we find the pressure at A is $1000 \cdot 10 \cdot 1.5 = 15000$ Pa.

7. A cylinder has a radius R and weight G. It rolls without slipping towards the step as shown below. For the cylinder-step system, which of the following quantities are *not* conserved during the collision?



(A) Linear Momentum

(B) Angular Momentum

(C) Energy $\leftarrow$ Correct

(D) All of the above are conserved.

(E) At least two of the above are not conserved.

Solution: We look at each of the answer choices, (A), (B), and (C) and see which are not conserved and which are.

1. Linear Momentum: Linear momentum is conserved during the collision because the step is fixed to the ground. If it was not, then linear momentum would not be conserved as the step will be moving in an inelastic collision with the cylinder and losing kinetic energy.

2. Angular Momentum: Angular momentum with respect to the edge is conserved during the collision but not after.

3. Energy: Energy is not conserved during any collision (look at 1), however energy is conserved after.

Thus we can see that only energy is not conserved during the collision.

8. Alice and Bob each hold an end of a spring of spring constant k and very small mass δm. Alice pulls the left end of the spring leftward with a force of magnitude F_A, while Bob pulls the right end rightward with a force of magnitude F_B, where $F_A > F_B$. What happens to the spring?

(A) It is stretched by a length of $\frac{F_A}{k}$; its center of mass moves leftward with acceleration $\frac{F_A - F_B}{\delta m}$.

(B) It is stretched by a length of $\frac{F_B}{k}$; its center of mass moves leftward with acceleration $\frac{F_A - F_B}{\delta m}$. ← Correct

(C) It is stretched by a length of $\frac{F_A + F_B}{2k}$; its center of mass does not move.

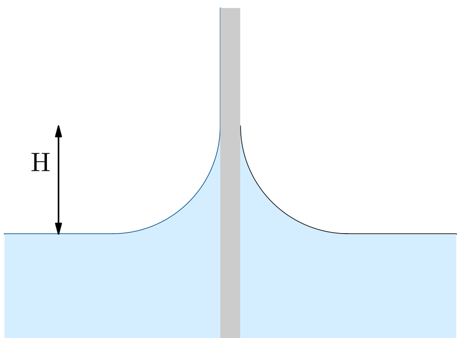
(D) It is stretched by a length of $\frac{F_A + F_B}{k}$; its center of mass moves leftward with acceleration $\frac{F_A - F_B}{\delta m}$.

(E) It is stretched by a length of $\frac{F_A - F_B}{k}$; its center of mass moves leftward with acceleration $\frac{F_A + F_B}{2\delta m}$.

Solution: By the nature of a spring, the spring force is just the tension that stretches the spring. For a spring to have tension T, both sides of the spring must be pulled at a force T. For instance, when a spring is attached to a wall and pulled by someone with force T, it is pulled back by the wall with equal and opposite force T to remain attached, so it assumes tension T. In this case, imagine what would happen if both Alice and Bob pulled with force F_B. As mentioned before, the spring would just assume tension F_B. But now suppose Alice increases her force by some difference $F_A - F_B$. Though the tension in the spring remains at F_B (since Bob keeps his force constant), the spring (particularly its center of mass) will move leftward with force $F_A - F_B$. We know, therefore, that $F_B = kx$, where x is the length by which the spring is stretched, so $x = \boxed{\frac{F_B}{k}}$. And the center of mass must accelerate at a, where

$$a\delta m = F_A - F_B \implies a = \boxed{\frac{F_A - F_B}{\delta m}}.$$

9. A smooth vertical wall has been thoroughly wetted by water as shown in the figure below. The water clings to the wall due to surface tension, and therefore the water level increases by a height H. Now, the temperature of the surroundings increases, in turn making the density of the water decrease by 3% and the surface tension increase by 0.8%. Approximately, by what percent does the height of the water increase/decrease?



(A) 2% lower

(B) 3% lower

(C) 1% higher

(D) 2% higher ← Correct

(E) 3% higher

Solution: We can try to find a formula for the height H in terms of the surface tension σ, density ρ, and radius r. We know that

$$[H] = L$$
$$[\sigma] = MT^{-2}$$
$$[\rho] = ML^{-3}$$
$$[g] = LT^{-2}.$$

Suppose we have

$$H = c\sigma^{\alpha}\rho^{\beta}r^{\gamma}.$$

Then

$$[H] = (MT^{-2})^{\alpha}(ML^{-3})^{\beta}(LT^{-2})^{\gamma} = M^{\alpha+\beta}L^{-3\beta}T^{-2\alpha+\gamma}.$$

Comparing exponents, we have

$$\begin{aligned} \alpha + \beta &= 0 \\ -3\beta &= 1 \\ -2\alpha + \gamma &= 0. \end{aligned}$$

This gives the solution $(\alpha, \beta, \gamma) = (1, -1, -1)$. Therefore, $H = c\sqrt{\frac{\sigma}{\rho r}}$. We now need to analyze the percent difference in height when the density has decreased by 2% and the surface tension has increased by 0.3%. We have

$$H' = c\sqrt{\frac{1.02\sigma}{0.997\rho r}} \approx 1.02H.$$

Thus the height has increased by 2%.

Note: We can also roughly equate the gravitational force to the surface tension force to result in

$$\rho H^3 g \sim \sigma H.$$

Solving for H, we find

$$H \sim \left(\frac{\sigma}{\rho g}\right)^{1/2}.$$

This is the only dimensionally-correct combination of σ, ρ, and g to give a length.

10. An unstretchable string of length ℓ is connected to a lead sphere and hanged to the top of a cylindrical container a distance r from the center. The container rotates about its axis with an angular velocity ω which makes the pendulum create an angle θ to the horizontal. Now let the container fill up with water and yet once again rotate about its axis with an angular velocity ω. Which of the following must be true?

(A) The string rotates to an angle less than θ and has the same tension before.

(B) The string rotates to the same angle θ but has lower tension tension than before. $\leftarrow$ Correct

(C) The string rotates to the same angle θ and has equivalent tension as before.

(D) The string rotates to an angle greater than θ and has a lower tension than before.

(E) The string rotates to an angle greater than θ and has higher tension than before.

Solution: Without the fluid, the effective gravity is the vector sum of $m\vec{g} + m\omega^2\vec{r}$. The rope will be in the same line as the effective gravity.

When submerged in a fluid, the buoyant force acts opposite to the effective gravity. However, if we use rotate at an angular velocity ω, the effective gravity should still be the same. Therefore the buoyant force will also act along the rope, contributing zero torque. Therefore, although the tension decreases the angle will still be the same.

11. You are holding a baseball bat of mass m and length ℓ at its handle on the bottom, which is a distance d above the ground, in vertical position, assuming ideal conditions. You swing the bat through an angle θ at an angular acceleration α clockwise and then release it. The bat hits the ground with total energy

(A) $\frac{1}{3}m\ell^2\alpha\theta + mgd$.

(B) $\frac{1}{12}m\ell^2\alpha\theta + mg\,(d + \ell\cos\theta)$.

(C) $\frac{1}{3}m\ell^2\alpha\theta + mg\left(d + \frac{\ell}{2}\cos\theta\right)$. $\leftarrow$ Correct

(D) $m\ell^2\alpha\theta + mg\left(d + \frac{\ell}{2}\right)$.

(E) unknown, as there is not enough information.

Solution: First, you hold the bat firmly, and it has gravitational potential energy $mg\left(d + \frac{\ell}{2}\right)$. Now, you apply an energy $\tau\theta$, where τ is the torque you apply and θ is the angle through which you swing. We see that $\tau = I\alpha$. The value of I for a rod pivoted about one end is $\frac{1}{3}m\ell^2$. Overall, this is just $\frac{1}{3}m\ell^2\alpha\theta$.
After finding the new gravitational potential energy through basic trigonometric methods by accounting for the difference in height, we find a total energy of $\boxed{\frac{1}{3}m\ell^2\alpha\theta + mg\left(d + \frac{\ell}{2}\cos\theta\right)}$.

12. The Little Prince is on a new planet with a grandfather clock calibrated on Earth. This new solar system is an exact copy of our solar system except all distances have been doubled. For example, the distance from the planet to the star will be doubled, and the radius will be doubled as well, among other lengths. The density stays the same. How many Earth years will the Little Prince measure to be one year on this new planet?

(A) 1/2 Earth years

(B) $1/\sqrt{2}$ Earth years

(C) $\sqrt{2}$ Earth years $\leftarrow$ Correct

(D) 2 Earth years

(E) $2\sqrt{2}$ Earth years

Solution: First, let us determine the period of the orbit. We have:

$$\omega^2 R = \frac{GM}{R^2}$$
$$\left(\frac{2\pi}{T}\right)^2 = \frac{G\rho\frac{4}{3}r^3}{R^3}$$

Notice that if r and R both doubled then the right hand side wouldn't change, meaning the period of one year on Htrae is the same as one year on Earth.
Next, let us determine the gravitational acceleration on Htrae. We have:

$$\begin{aligned} g_{\text{new}} &= \frac{GM}{R_{\text{new}}^2} \\ &= \frac{G\rho\frac{4}{3}R_{\text{new}}^3}{R_{\text{new}}^2} \\ &= \frac{4}{3}G\rho R_{\text{new}} \\ &= \frac{4}{3}G\rho(2R) \\ g_{\text{new}} &= 2g \end{aligned}$$

period of the pendulum:

$$\begin{aligned} T_{\text{new}} &= 2\pi\sqrt{\frac{\ell}{g_{\text{new}}}} \\ &= 2\pi\sqrt{\frac{\ell}{2g}} \\ T_{\text{new}} &= \frac{T}{\sqrt{2}} \end{aligned}$$

where T is the period of the pendulum if it was on Earth. If every period made by the pendulum corresponded with one second, then one second "measured" on Htrae will actually be $\frac{1}{\sqrt{2}}$ real seconds as measured with an accurate clock. Thus, the "measured" time for one year will be $\sqrt{2}$ Earth years

13. Bronze is an alloy of copper and tin. In a certain batch of bronze, the copper used has a density of 9.32 ± 0.14 g/cm^3. The tin has a density of 6.56 ± 0.81 g/cm^3. The alloy is $78.00 \pm 0.02\%$ copper by volume. If all of the uncertainties are independent, what is the uncertainty in the density of the resulting bronze?

(A) 0.09 g/cm^3

(B) 0.12 g/cm^3

(C) 0.17 g/cm^3

(D) 0.21 g/cm^3 ← Correct

(E) 0.28 g/cm^3

Solution: The density of the alloy is a weighted average of the densities of the components. If the fraction of volume that is copper is α, then the density of the alloy is

$$\rho = \alpha\rho_c + (1-\alpha)\rho_z.$$

The multiplication by factors α and $(1-\alpha)$ are examples of scaling a random variable. They do not change the relative error, and they multiply the absolute error by α and $(1-\alpha)$ respectively. Using addition in quadrature, the total error is

$$\sigma_\rho = \sqrt{(\alpha(\Delta\rho_c))^2 + ((1-\alpha)(\Delta\rho_z))^2}$$

Evaluating with the numbers given in the problem, the result is $\Delta\rho = \boxed{0.21 \text{ g/cm}^3.}$

14. A rope of uniform density λ moves under its own momentum at a constant velocity v. It approaches an incline at angle θ and moves all the way up until half of it's length is on the incline. If the total length of the rope is L, then how much time elapses from the moment the rope begins to go up the incline until it stops momentarily? Assume the rope remains taut throughout the entire process.

(A) $\frac{\pi}{4}\left(\frac{v}{g\sin\theta}\right)^{1/3}$

(B) $\frac{v}{g\sin\theta}$

(C) $\frac{vg\sin\theta}{2L^2}$

(D) $\frac{\pi}{2}\sqrt{\frac{L}{g\sin\theta}}$ ← Correct

(E) $2\pi\sqrt{\frac{L}{g\sin\theta}}$

Solution: Let us choose a coordinate system with the origin at the foot of the incline and the x-axis pointing up the incline. If $\lambda = M/L$ is the density of the train and x is the length of the train that is on the incline, the mass of the part of the train on the incline is λx. Then Newton's second law provides

$$Ma = -\lambda x g\sin\theta = -\frac{M}{L}xg\sin\theta$$

this means

$$a = -\frac{g\sin\theta}{L}x$$

This equation resembles the motion of a harmonic oscillator. (see footnote) The period T of oscillations is

$$T = 2\pi\sqrt{\frac{L}{g\sin\theta}}$$

The time for the rope to stop will be a quarter of this period or,

$$\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{L}{g\sin\theta}}$$

Footnote: For some of you who do not see why this represents the equation of a harmonic oscillator think of the equation

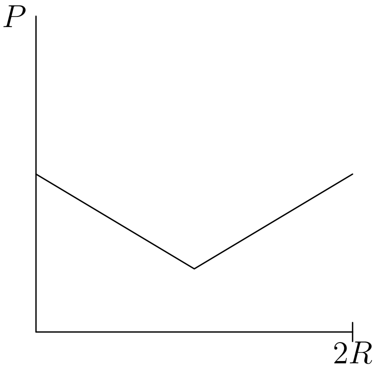
$$F = -kx$$

From here it should be easy to see that $\frac{k}{m} = \frac{g\sin\theta}{L}$ which allows us to do our calculations from henceforth.

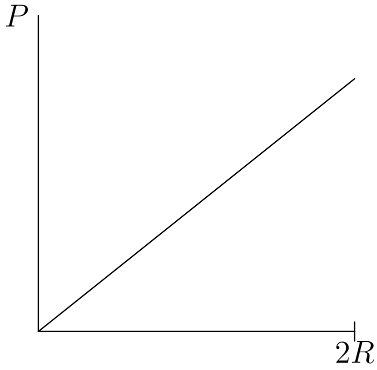
Remark: The initial momentum of the rope has no effect on the time it takes to get up to the ramp!!

15. In deep space, a large cylindrical spaceship with radius R is rotating with an angular velocity ω. A drone takes off from the ground and in the frame of the spacecraft, flies directly upwards with a constant velocity to the other side a distance $2R$ away. In the frame of the rotating spaceship, which of the following shows the power P of the drone as a function of distance?

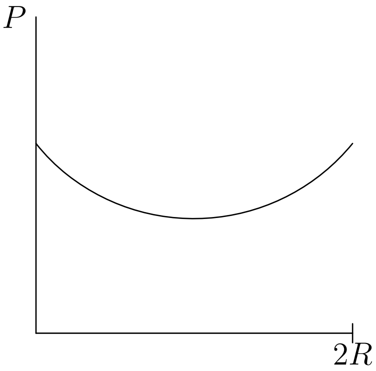
(A)



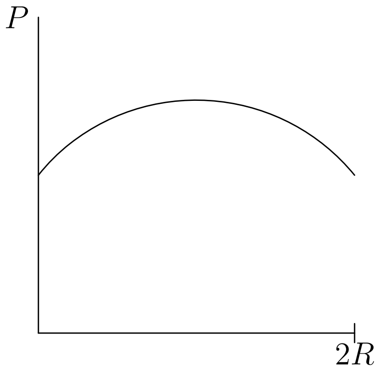
(B)



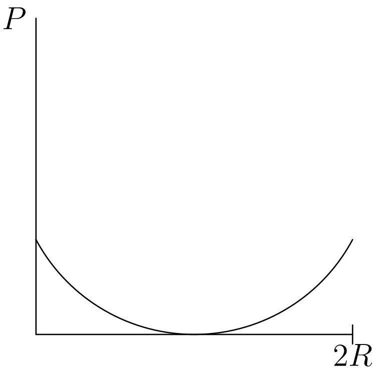
(C) ← Correct



(D)



(E)



Solution: The force required to move at constant velocity is given by:

$$F = \sqrt{F^2_{\text{coriolsis}} + F^2_{\text{centrifugal}}} = \sqrt{4m^2\omega^2v^2 + m^2\omega^4R^2}$$

thus the power would be:

$$P = m\omega v\sqrt{4v^2 + \omega^2R^2}$$

Plugging this into a graphing calculator, or using other similar means, you will get the answer of C.

Footnote: Some of you may not understand why the graph never touches zero. This is because you thought the Coriolis force does no work since it is perpendicular to the Centrifugal force. However, this reasoning is invalid because the Coriolis force is perpendicular to velocity, so technically the Coriolis force technically doesn't do any work – but an actual, real flying drone would still need to spend energy to counter it, e.g. by pushing air in the opposite direction.

16. In xyz space, all coordinates are given in meters. A rectangle with uniform density is constructed with vertices at coordinates $(0, 0, 0)$, $(5, 0, 0)$, $(5, 4, 0)$, and $(0, 4, 0)$. Next, the quarter circle bounded by the arc between $(2, 0, 0)$ and $(0, 2, 0)$ centered at $(0, 0, 0)$ is cut out from the rectangle. If the remaining figure has mass 3 kg, what is its moment of inertia about the line passing through $(5, 4, 0)$ and parallel to the z-axis?

    (A) $10\ \mathrm{kg \cdot m^2}$

    (B) $20\ \mathrm{kg \cdot m^2}$ ← Correct

    (C) $30\ \mathrm{kg \cdot m^2}$

    (D) $40\ \mathrm{kg \cdot m^2}$

    (E) $50\ \mathrm{kg \cdot m^2}$

Solution: Firstly, we must find the center of mass of the quarter circle that is removed. Let's suppose we rotate a quarter-circle so that it becomes a half-sphere (dome-shape). The volume of the half-sphere is:

$$V = \frac{2}{3}\pi R^3$$

The area of the figure we rotated is:

$$A = \frac{1}{2}\pi R^2$$

The distance the centroid moved during the rotation is d. Pappus's Theorem tells us that:

$$V = Ad$$

Solving for d gives:

$$d = \frac{4R}{3}$$

However, this only tells us how much the centroid moved, not where it is. But notice that since it moves in a circle, we have:

$$2\pi r = \frac{4R}{3} \implies r = \frac{2R}{3\pi}$$

This gives the 'horizontal' distance from the corner to the centroid. The absolute distance would be:

$$D = \frac{2\sqrt{2}R}{3\pi}.$$

We now try to find the moment of inertia of the rectangle itself. We can imagine the rectangle to be composed of three rods. The moment of inertia then about its end is

$$I = \frac{1}{3}(L_x^2 + L_y^2).$$

The moment of inertia of the rectangle at $(5, 4, 0)$ is:

$$\frac{1}{3}M(5^2 + 4^2) = 13.67M$$

The moment of inertia of the quarter circle at is:

$$-\left(\frac{1}{2}m(\frac{2\sqrt{2}R}{3\pi}^2) + m(4^2 + 5^2)\right) = m\left(\frac{4\sqrt{2}}{3\pi^2} + 41\right)$$

The mass of the quarter circle is:

$$\frac{0.25\pi(2)^2}{20} = \frac{M-3}{M} \implies M = 3.56$$

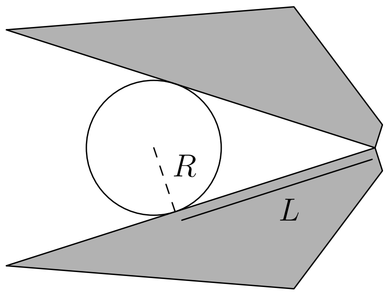
and

$$m = 0.56$$

Plugging these values in gives:

$$I = 13.67(3.56) - 22.38 = 22.387$$

17. A clamp is used to pick up a gumball with mass m and radius R. The contact point is a distance L from the pivot. Then the clamp is held parallel to the page. The following diagram is a top-down view.



Given that the coefficient of friction is μ, what is the minimum normal force each arm needs to exert on the gumball?

(A) $\dfrac{mg}{\sqrt{\mu^2 - 1}}$

(B) $\dfrac{mg}{\sqrt{\mu^2 + (L/R)^2}}$

(C) $\dfrac{mg}{2\sqrt{\mu^2 - (L/R)^2}}$

(D) $\dfrac{mg}{2\sqrt{\mu^2 - (R/L)^2}}$ ← Correct

(E) The force of friction will increase as the normal force is increased, so it can withstand an infinite normal force.

Solution: There are two components to friction, one in the vertical direction f_y, and one in the direction along the chopstick f_x. The vertical component of friction support the weight of the gumball. We have:

$$2f_y = mg$$

The 2 is there because there are two chopsticks. Next, consider the normal force. There are two components to the normal force as well. One component going leftwards (refer to figure above) and one component going vertically parallel to page. The horizontal component is given by $N\sin\theta$ and the vertical component is given by $N\cos\theta$. We don't have to worry about the vertical component because it will cancel out with the vertical component from the other chopstick. However, the friction force has to be sufficient to balance the horizontal component of the normal force. We have:

$$f_x \cos\theta = N\sin\theta \implies f_x = N\tan\theta$$

At the maximum, the total friction force is:

$$\mu N = \sqrt{f_x^2 + f_y^2} = \sqrt{\frac{m^2g^2}{4} + N^2 \tan^2\theta}$$

Solving for N gives:

$$\frac{mg}{2\sqrt{\mu^2 - \tan^2\theta}} = \boxed{\frac{mg}{2\sqrt{\mu^2 - (R/\ell)^2}}}$$

Notice that if $\mu = R/\ell$, the gumball can withstand an infinite normal force.

18. A block of mass m is attached to a rope of initial length L, initial cross-sectional area A, and Young's modulus E. The rope stretches so that the volume stays constant. What is the stretch ΔL necessary in order to maintain static equilibrium when hung vertically?

    Note: the Young's modulus E is defined such that, for any force F, $F = AE\frac{\Delta L}{L}$.

    (A) $\dfrac{mgL}{AE}$

    (B) $\dfrac{mgL}{AE - mg}$ ← Correct

    (C) $\dfrac{mgL}{AE - 2mg}$

    (D) $\dfrac{2mgL}{AE - mg}$

    (E) $\dfrac{2mgL}{AE - 2mg}$

Solution: The tension in the rope is defined as:

$$T = AE\frac{\Delta L}{L}$$

At first, we might set this equal to mg but we can't do this! The question states that the volume of the rope stays constant. This means as ΔL increases, A will decrease. We must have the constraint:

$$AL = a(L + \Delta L) \to a = \frac{AL}{L + \Delta L}$$

substituting this in gives:

$$T = \frac{A}{L + \Delta L} E \Delta L$$

Setting it equal to mg gives:

$$\Delta L = \frac{mgL}{AE - mg}$$

Note: This problem is very tricky! In fact, some of you may be wondering why your solution is incorrect. One incorrect solution can look sort of like:

The tension in the wire is defined as

$$T = PA = AE\frac{\Delta L}{L}$$

This means that the wire can be modeled as a spring with spring constant (see footnote)

$$k = \frac{T}{\Delta L} = E\frac{A}{L}$$

Now we have simplified the problem to be:
A block of mass m is attached to a spring of initial length L and spring constant $k = E\frac{A}{L}$. The block is let go from rest and undergoes oscillations. What is the difference in height between the topmost and bottom most point?
This is a very simple problem now. By a simple force analysis, the force of gravity is equal to the restoring force $k\Delta L$. Thus the difference in the topmost and bottom most point is

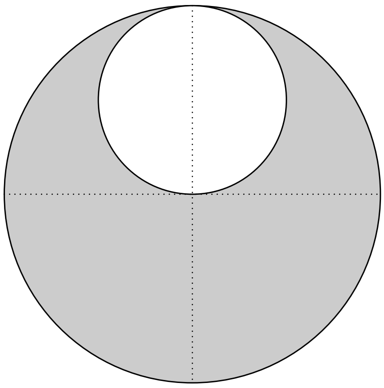
$$\Delta L = \frac{mg}{k} = \frac{mg}{E\frac{A}{L}} = \frac{mgL}{EA}$$

Footnote: The reason that this can be modeled as a spring is because when the mass is distance ΔL down it recieves a restoring force of $k\Delta L$. If we look at what we have for our spring constant $E\frac{A}{L}$ and multiply it with ΔL we get $F = k\Delta L = AE\frac{\Delta L}{L}$ which is what Young's modulus is defined as.

However, this solution is incorrect!! This is because we neglected that volume has to stay constant. Can you see why volume does not remain constant in the bogus solution provided above?

Use the following information for problems 19 and 20.

A hole of radius $R/2$ is cut out of a uniformly solid disk of radius R and mass M.



19. The object is slightly pushed so that it oscillates with respect to time. Find the period of oscillations given that there is sufficient friction to prevent the disk from slipping.

(A) $2\pi\sqrt{\frac{20R}{67g}}$

(B) $2\pi\sqrt{\frac{80R}{87g}}$

(C) $2\pi\sqrt{\frac{24R}{20g}}$

(D) $2\pi\sqrt{\frac{5R}{3g}}$

(E) $2\pi\sqrt{\frac{29R}{20g}}$ ← Correct

Solution: Balance of torques gives us:

$$I\ddot{\theta} = -mgx_{\text{cm}}\theta$$

We'll break this up into a few parts.
To find the moment of inertia of the gray disk around the contact point, we have:

$$\frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

The moment of inertia of the "hole" around the contact point is:

$$\frac{1}{2}\left(\frac{-M}{4}\right)\left(\frac{R}{2}\right)^2 + \left(\frac{-M}{4}\right)\left(\frac{3R}{2}\right)^2 = -\frac{19}{32}MR^2$$

so the total moment of inertia is:

$$I = \frac{29}{32}MR^2$$

To find the center of mass, we use the standard equation:

$$\begin{aligned} x_{\text{cm}} &= \frac{M(R) + \left(\frac{-M}{4}\right)\left(\frac{3R}{2}\right)}{M + \left(\frac{-M}{4}\right)} \\ &= \frac{\left(\frac{5R}{8}\right)}{\left(\frac{3}{4}\right)} \\ &= \frac{5R}{6} \end{aligned}$$

To find the final mass of the object, we have:

$$m = M - \frac{M}{4} = \frac{3M}{4}$$

Plugging everything in:

$$\begin{aligned} \left(\frac{29}{32}MR^2\right)\ddot{\theta} &= -\left(\frac{3M}{4}\right)g\left(\frac{5R}{6}\right)\theta \\ \frac{29}{20}R\ddot{\theta} &= -g\theta \end{aligned}$$

so we have:

$$\omega^2 = \frac{20g}{29R}$$

or

$$\boxed{T = 2\pi\sqrt{\frac{29R}{20g}}}$$

20. A horizontal force is delivered to the top of the contraption in a very short period of time. What is the minimum impulse that needs to be delivered to make the disk rotate one full revolution? Assume it rolls without slipping.

(A) $M\sqrt{\frac{1}{12}gR}$

(B) $M\sqrt{\frac{64}{13}gR}$

(C) $M\sqrt{\frac{13}{256}gR}$

(D) $M\sqrt{\frac{27}{13}gR}$

(E) $M\sqrt{\frac{36}{13}gR}$ $\leftarrow$ Correct

Solution: The change in potential energy from going from top to bottom will be

$$\Delta U = 2\left(\frac{M}{4}\right)g\left(\frac{R}{2}\right) = \frac{1}{4}MgR$$

by conservation of energy $\Delta U = \frac{1}{2}I_{\text{center}}\omega^2$. To find I_{center} we use parallel axis theorem

$$I_{\text{center}} = \frac{1}{2}MR^2 - \frac{1}{2}\left(\frac{M}{4}\right)\left(\frac{R}{2}\right)^2 + \left(-\frac{M}{4}\right)\left(\frac{R}{2}\right)^2 = \frac{13}{32}MR^2$$

Going back to our conservation of energy equation yields

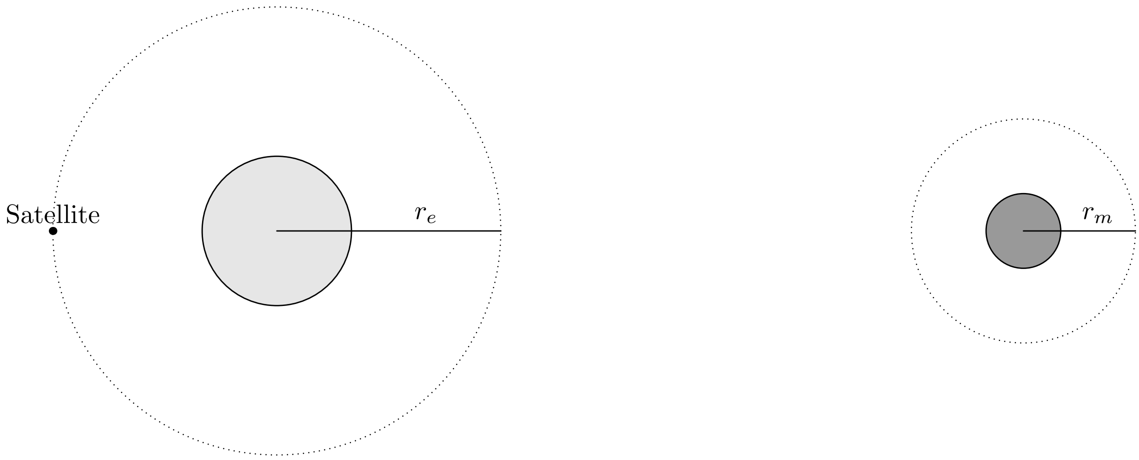
$$\frac{1}{2}\left(\frac{13}{32}MR^2\right)\omega^2 = \frac{1}{4}MgR$$

$$v^2 = \frac{64}{13}gR \implies \Delta v = 8\sqrt{\frac{gR}{13}}.$$

The impulse J will now simply be $m\Delta v$ thus by substiuting we have

$$J = \left(\frac{3M}{4}\right)\Delta v = 6M\sqrt{\frac{gR}{13}} = \boxed{M\sqrt{\frac{36}{13}gR}}.$$

21. NASA wishes to bring a small rocket of mass m to the moon. First, it starts off in a circular orbit at a semi-major axis r_e around Earth. The engines fire in a negligible amount of time and bring the rocket to the moon, a distance d away. You may assume that $d \gg r_e$. Once the rocket reaches the moon, the engines fire once again and bring the rocket into a circular orbit at a semi-major axis r_m. The mass of the Earth and the moon are M_e and M_m respectively.



What is the total impulse the engines must impart on the rocket?

(A) $m\left(\sqrt{2}-1\right)\left(\sqrt{\frac{GM_e}{r_e}} + \sqrt{\frac{GM_m}{r_m}}\right)$ $\leftarrow$ Correct

(B) $m\left(\sqrt{2}-1\right)\left(\sqrt{\frac{GM_e}{r_e}}\right) + m\sqrt{\frac{GM_m}{r_m}}$

(C) $m\left(\sqrt{\frac{GM_e}{r_e}}+\sqrt{\frac{GM_m}{r_m}}\right)$

(D) $m\left(\sqrt{\frac{2GM_e}{r_e}}+\sqrt{\frac{2GM_m}{r_m}}\right)$

(E) $m\left(\sqrt{\frac{GM_e}{r_e}}+\sqrt{\frac{2GM_m}{r_m}}\right)$

Solution: Initially the rocket is in a circular orbit around Earth. It has an orbital speed of

$$v_1 = \sqrt{\frac{GM_e}{r_e}}$$

Let d be the distance between Earth and the moon. After performing a burn, the rocket should not have an elliptical orbit where the furthest point is roughly at an altitude d. Realistically, it would be $d + r_e + r_m$ but we can assume $r_e \ll d$ and $r_m \ll d$. Thus the semi-major axis will now be $d/2$. Per the vis-visa equation, the speed will now be:

$$v_2 = \sqrt{GM_e\left(\frac{2}{r_e}-\frac{2}{d}\right)}$$

Substituting in values, we can calculate $\Delta v_1 = v_2 - v_1$ Now let us determine the speed of the spacecraft once it has reached the moon. Conservation of energy gives:

$$\begin{aligned}\frac{1}{2}v_3^2 - \frac{GM_m}{r_m} - \frac{GM_e}{d} &= \frac{1}{2}v_2^2 - \frac{GM_e}{r_e}\\ v_3^2 &= GM_e\left(\frac{2}{r_e}-\frac{2}{d}\right) - \frac{2GM_e}{r_e} + \frac{2GM_m}{r_m} + \frac{2GM_e}{d}\\ &= \frac{2GM_m}{r_m}\end{aligned}$$

so we have

$$v_3 = \sqrt{\frac{2GM_m}{r_m}}$$

Now we need to circularize the orbit. After circularizing, the speed will then be:

$$v_4 = \sqrt{\frac{GM_m}{r_m}}$$

so the change in speed will be:

$$\Delta v_2 = v_3 - v_4 = \left(\sqrt{2}-1\right)\sqrt{\frac{GM_m}{r_m}}$$

so the total change in velocity the engines need to impart (impulse/mass) is:

$$\Delta v_1 + \Delta v_2 = \sqrt{GM_e\left(\frac{2}{r_e}-\frac{2}{d}\right)} - \sqrt{\frac{GM_e}{r_e}} + \left(\sqrt{2}-1\right)\sqrt{\frac{GM_m}{r_m}}$$

We can make this easier if we let them assume d is very big such that:

$$\Delta v_T = \left(\sqrt{2}-1\right)\left(\sqrt{\frac{GM_m}{r_m}}+\sqrt{\frac{GM_e}{r_e}}\right)$$

Note: this solution shows the method to solve this in the general case. However, since the problem stated that d is very large, we can say that as the rocket performs the burn to intercept with the moon, it creates an elliptical orbit where the speed at aphelion is zero and the potential energy is also zero. Thus we can use energy conservation:

$$-\frac{GM_e m}{r_e^2} + \frac{1}{2}mv^2 = 0 \implies v = \sqrt{\frac{2GM_e}{r_e^2}}$$

22. Two glass balls of equal mass are at rest on an infinite track. There is a distance d between the two balls. At time $t = 0$, Ball 1 is given a constant force F to the right. After every collision, the force F on the first ball is multiplied by 2. If all collisions are elastic, after a large number of collisions, what is the sum of the distances ball 1 travels between subsequent collisions?

(A) $10d$

(B) $\frac{11}{2}d$

(C) $8d$ ← Correct

(D) $\frac{29}{3}d$

(E) Much greater than any of the above

Solution: The speed of ball A as it collides with ball B the first time is given by:

$$v = \sqrt{\frac{2Fd}{m}}$$

After they collide, ball A will initially be stationary and ball B will be travelling rightwards at a speed v. We can switch reference frames and say that ball B is stationary and ball A is travelling leftwards but experiencing a force $2F$. The time until the next impact is given by:

$$vt - \frac{2F}{2m}t^2 = 0$$
$$t_1 = \frac{2mv}{2F}$$

Ball B will have travelled a distance $d_1 = v\left(\frac{mv}{F}\right)$

Next, their relative speeds upon impact is going to be v per conservation. A net work of zero is done by moving ball A leftwards and then rightwards again to the same spot (since ball B is stationary). Therefore in the lab frame right before the collision, ball A will be moving rightwards with speed $2v$ and ball B will be moving rightwards with a speed v. In an elastic collision between two equal masses, their velocities flip after colliding. Therefore after the collision ball A will be moving right at a speed of v and ball B will be moving right at a speed of $2v$.

This is the most important part. Notice that again if we switch into the frame of ball B, we have ball A moving leftwards with a speed v. As with before, their relative velocities at impact will also be v. Since this is similar to the first scenario, we can use the same results:
We have:

$$t_2 = \frac{2mv}{2^2F}$$

and

$$d_2 = 2v\left(\frac{2mv}{2^2F}\right)$$

We can generalize it to:

$$d_n = nv\left(\frac{2mv}{2^nF}\right) = \frac{2mv^2}{F}\left(\frac{n}{2^n}\right)$$

$$\sum_{n=0}^{\infty}\frac{2mv^2}{F}\left(\frac{n}{2^n}\right) = \frac{2mv^2}{F}\sum_{n=0}^{\infty}\frac{n}{2^n}$$

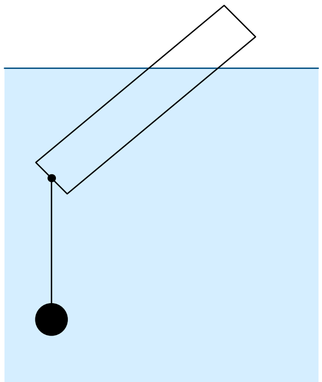
We can calculate the summation by telescoping: Let $S = \sum_{n=0}^{\infty}\frac{n}{2^n}$.

$$S = \sum_{n=0}^{\infty}\frac{n}{2^n} \implies S = \frac{1}{2} + \frac{1}{2}\left(\frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \cdots\right) \implies S = \frac{1}{2} + \frac{1}{2}\left(S + \sum_{n=1}^{\infty}\frac{1}{2^n}\right)$$

$$\implies S = \frac{1}{2} + \frac{1}{2}(S+1) \implies S = 2$$

Therefore, the final answer is $\frac{4mv^2}{F} = \boxed{8d}$ (because $v^2 = \frac{2Fd}{m}$).

23. A thin homogeneous cylindrical float is made out of a light substance with a density 0.5 g/cm^3. A lead sinker of density 204.53 g/cm^3 is tied with fishing line to the bottom of the float. Let the mass of the float be m, and the mass of the sinker be M. What conditions must the ratio of the masses of the sinker and the float satisfy for the float to rest vertically in the water? (Neglect the forces of surface tension. The density of water is 1 g/cm^3).



(A) $0.34 \leq \frac{M}{m} \leq 1.34$

(B) $0.41 \leq \frac{M}{m} \leq 1$ ← Correct

(C) $0.49 \leq \frac{M}{m} \leq 0.76$

(D) $0.57 \leq \frac{M}{m} \leq 1.53$

(E) $1.21 \leq \frac{M}{m} 1.73$

Solution: The buoyant force is equal to the weight of the displaced water given by $F_b = \rho_0 S \ell_1 g$, where S is the cross sectional area of the float and ℓ_1 is the length of the submerged portion of the float. We find, from the mass of the float, that $S = \frac{m}{\rho_1 \ell}$ where ℓ is the total length of the float. Combining these two relationships gives

$$F_b = mg\frac{\rho_0}{\rho_1}\frac{\ell_1}{\ell}$$

The float will float if the buoyant force exerted by the float balances the weight of the float and the force F exerted by the sinker. Since the sinker is in the water, this force is given by

$$F + mg = F_b$$

we also know the total force given by the sinker is equal to

$$F = Mg - \rho_0 \frac{M}{\rho_2} g = Mg\left(1 - \frac{\rho_0}{\rho_2}\right)$$

Therefore, F_b is also equal to

$$F_b = mg + Mg\left(1 - \frac{\rho_0}{\rho_2}\right)$$

Comparing these two expressions for F_b yields,

$$\frac{\ell_1}{\ell} = \frac{\rho_1}{\rho_0}\left[1 + \frac{M}{m}\left(1 - \frac{\rho_0}{\rho_2}\right)\right]$$

Since $\ell_1 \leq \ell$ then

$$\frac{M}{m} \leq \frac{\frac{\rho_0}{\rho_1} - 1}{1 - \frac{\rho_0}{\rho_2}}.$$

In order for the float to sit in a vertical position, the torque produced by the buoyant force must be at least as large as the torque due to the float's weight:

$$\frac{F_b \ell_1 \sin\theta}{2} = \frac{mg\ell \sin\theta}{2}$$

Substituting F_b from our previous derivation, we get

$$mg\frac{\rho_0}{\rho_1}\frac{\ell_1}{\ell}\ell_1 \geq mg\ell$$

or,

$$\frac{\rho_0}{\rho_1} \geq \frac{\ell^2}{\ell_1^2}, \quad \frac{\ell_1}{\ell} \geq \sqrt{\frac{\rho_1}{\rho_0}}$$

Taking into account for the ratio $\frac{\ell_1}{\ell}$ and substituting into our previous expression gives us

$$\sqrt{\frac{\rho_1}{\rho_0}} \leq \frac{\rho_1}{\rho_0}\left[1 + \frac{M}{m}\left(1 - \frac{\rho_0}{\rho_2}\right)\right]$$

Which results in the second constraint of

$$\frac{M}{m} \geq \frac{\sqrt{\frac{\rho_0}{\rho_1}} - 1}{1 - \frac{\rho_0}{\rho_2}}.$$

Thus the conditions on the ratio of masses are

$$\frac{\sqrt{\frac{\rho_0}{\rho_1}} - 1}{1 - \frac{\rho_0}{\rho_2}} \leq \frac{M}{m} \leq \frac{\frac{\rho_0}{\rho_1} - 1}{1 - \frac{\rho_0}{\rho_2}}$$

Plugging these values in gives us our respective answer.

24. For all positive integers m, a block of mass $0.5^m \pm 0.2^m$ kg is placed at position m on the number line. The leftmost block is given velocity 1 ± 0.25 m/s rightward, and all collisions thereafter are inelastic. After all collisions occur, what will be the speed of the rightmost block, up to Gaussian uncertainty?

 (A) 0.5 ± 0.26 m/s $\leftarrow$ Correct

 (B) 0.5 ± 0.36 m/s

 (C) 0.5 ± 0.47 m/s

 (D) 0.5 ± 0.76 m/s

 (E) 0.5 ± 1.22 m/s

Solution: The velocity v_R of the rightmost block can be calculated as an infinite series; let m_i denote the mass of the i^{th} block:

$$v_R = \frac{m_1}{\sum_{i=0}^{\infty} m_i} v_1 = \frac{0.5}{\frac{0.5}{1-0.5}} = 0.5$$

This is justified by conservation of momentum. Now, we can calculate the uncertainty. Let $Q = \sum_{i=0}^{\infty} m_i = 1$. Therefore,

$$\delta Q = \sqrt{\delta m_1^2 + \delta m_2^2 + \cdots} = \sqrt{\sum_{k=1}^{\infty} \delta m_k^2} = \sqrt{\frac{0.2^2}{1-0.2^2}} \approx 0.204124.$$

Now, we may calculate the uncertainty in the velocity by utilizing the rule for multiplication of uncertainty:

$$\frac{\delta v_R}{v_R} = \sqrt{\left(\frac{\delta Q}{Q}\right)^2 + \left(\frac{\delta v_1}{v_1}\right)^2 + \left(\frac{\delta m_1}{m_1}\right)^2} \implies \delta v_R = v_R \sqrt{\left(\frac{\delta Q}{Q}\right)^2 + \left(\frac{\delta v_1}{v_1}\right)^2 + \left(\frac{\delta m_1}{m_1}\right)^2}$$

$$\implies \boxed{v_R = 0.50 \pm 0.26}$$

25. A cylindrical rod of length $\ell = 0.5$ m is held such that the bottom is a distance h above the surface of the water. The density of the rod is a fifth of that of water. At $t = 0$, the rod is dropped from rest and is just about able to be submerged completely before rising. How long does it take for the rod to reach its lowest point? Ignore effects from splashing and drag.

(A) $\sqrt{\frac{2\ell}{g}} + \pi\sqrt{\frac{\ell}{5g}}$

(B) $\sqrt{\frac{3\ell}{g}} + \pi\sqrt{\frac{\ell}{5g}}$ ← Correct

(C) $\sqrt{\frac{2\ell}{g}} + \pi\sqrt{\frac{3\ell}{g}}$

(D) $\sqrt{\frac{2\ell}{g}} + \frac{\pi}{2}\sqrt{\frac{3\ell}{g}}$

(E) $\sqrt{\frac{3\ell}{g}} + \frac{\pi}{2}\sqrt{\frac{\ell}{5g}}$

Solution: First, consider the forces acting upon the cylinder when it is partially submerged in water. When a length x is submerged, we have:

$$F_{\text{net}} = ma = mg - (2\pi r x)\rho_w g$$

However we also have:

$$\rho_w = 5\rho_{\text{cylinder}} = \frac{5m}{2\pi r \ell}$$

Substituting this in gives:

$$a = g - \frac{5g}{\ell} x$$

This is the equation of motion for simple harmonic motion with an angular frequency of:

$$\omega = \sqrt{\frac{5g}{\ell}}$$

The time for the cylinder to reach the bottomost point from when it first touches the water will be half of a period (minimum acceleration to maximum acceleration) or:

$$t_{\text{partially submerged}} = T/2 = \pi\sqrt{\frac{\ell}{5g}}$$

However, we also have to take into account the time for the object to drop a height h. To calculate h we can use the conservation of energy. We have already used a spring analog where

$$\frac{k}{m} = \frac{5g}{\ell} \implies k = \frac{5mg}{\ell}$$

and thus the potential energy stored through this interaction is

$$U = \frac{1}{2}\frac{5mg}{\ell}x^2$$

Conservation of energy gives:

$$\begin{aligned} mg(h+\ell) &= \frac{5mg}{2\ell}\ell^2 \\ h &= \frac{5}{2}\ell - \ell \\ &= \frac{3}{2}\ell \end{aligned}$$

And thus the time it took the cylinder to fall this distance is given by:

$$\frac{3}{2}\ell = \frac{1}{2}gt^2 \implies t_{\text{fall}} = \sqrt{\frac{3\ell}{g}}$$

The total time is thus:

$$t_{\text{fall}} + t_{\text{partially submerged}} = \sqrt{\frac{3\ell}{g}} + \pi\sqrt{\frac{\ell}{5g}}$$