PhODS MOCK 2021 PHYSICS COMPETITION

2021 $F = ma$ Exam

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet.
- Test under standard conditions, meaning that you must complete the test in 75 minutes in one sitting.
- This test contains 25 multiple choice questions. **Your answer to each question must be marked on the Google Forms answer sheet that accompanies the test.** Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- Correct answers will be awarded one point; incorrect answers and leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- All questions are equally weighted, but are not necessarily the same level of difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after Feburary 1, 2021.**

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

Harsh Deep, Victor Dumbrava, Ashmit Dutta, Proelectro, Aarjav Jain, QiLin Xue, Daniel Yang

1. A baseball player hits a baseball with a force F causing the ball to fly a distance d_0. If he hits the baseball again with twice the initial force $2F$ under the same identical conditions as before (i.e. the contact time between the bat and the ball remains the same), then what is the ratio of the new distance traveled d to the original distance travelled d_0?

   (A) $\sqrt{2}$
   (B) 2
   (C) $2^{3/2}$
   (D) 4 ← **CORRECT**
   (E) 8

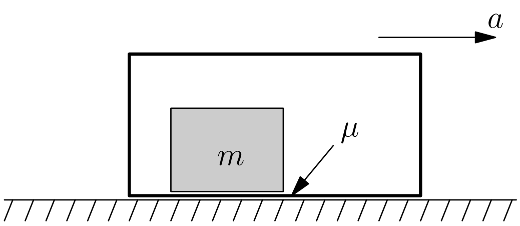
Solution: It is specified that the contact time between the bat and baseball remains the same which means that when an impulse of $2F$ is applied, $I = F\Delta t \implies I_{\text{new}} = 2I_0 \propto 2v$. Since the distance traveled, $d \propto v^2$, this means that $d_{\text{new}} = 4d_0$ which implies the ratio of distance travelled is 4.

2. A giant lilypad can be thought of as a large, uniform, circular cylinder of density 0.6 g/cm^3, thickness 4 mm, and radius 2 m. The lilypad is then placed in the middle of a large lake filled with water of density of 1000 kg/m^3. If a child is placed in the center of the lilypad, what is the maximum possible mass of the child (to the nearest 10 kg) such that the lilypad doesn't sink in the water?

   (A) 20 kg ← **CORRECT**
   (B) 30 kg
   (C) 40 kg
   (D) 60 kg
   (E) 80 kg

Solution: To get the largest possible weight we can say that the lilypad should be on the verge of sinking or would be fully submerged. Again, writing statics equation for the lilypad, $\rho_l V + m = \rho_w V$ $m = 400\left(\pi \cdot 2^2 \cdot 4 \cdot 10^{-3}\right)$ which is $m \approx 20.1$ kg

3. A block of mass $m = 10$ kg is placed in a cart of mass 20 kg, which is kept on a frictionless table as shown in the image below. The coefficient of friction between the cart and the block is $\mu = 0.3$. Initially, both objects are rest, and at time $t = 0$, the cart is suddenly pulled by some external force with constant acceleration $a = 5$ m/s^2. Find the maximum non-impulsive force applied by the cart on the block when the block is at rest with respect to the cart. The answer choice may not be exact.

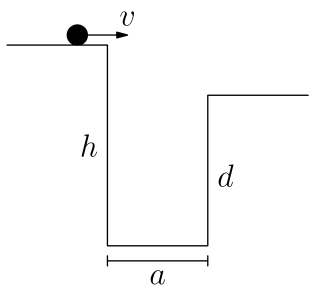


   (A) 30 N
   (B) 50 N
   (C) 100 N
   (D) 110 N ← **CORRECT**
   (E) 140 N

Solution: Maximum force will be applied when the blocks comes to the left end of the cart. Net horizontal force of both friction and normal is ma and net vertical force on block by cart is mg.

$$\Rightarrow F_{\text{max}} = m\sqrt{g^2 + a^2} \approx 110 \text{ N}$$

4. A ball flies off a small rectangular ditch with a height h (on the left side) and a height d (on the right side) with a width a as shown below. How much velocity does the ball need such that it lands on the other side of the ditch?



(A) $v > a\sqrt{\dfrac{g}{4(h-d)}}$

(B) $v > a\sqrt{\dfrac{g}{2(h-d)}}$ ← **CORRECT**

(C) $v > a\sqrt{\dfrac{g}{h-d}}$

(D) $v > a\sqrt{\dfrac{2g}{h-d}}$

(E) $v > a\sqrt{\dfrac{4g}{h-d}}$

Solution: Consider the two kinematic equations in the x and y directions:

$$v = at \implies t = \frac{v}{a}$$

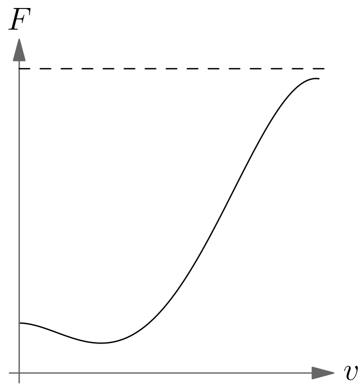
$$h - d = \frac{1}{2}gt^2$$

Substituting equation 1 into equation 2 yields

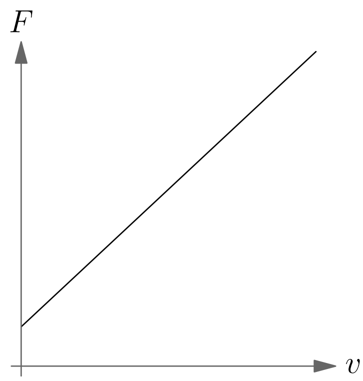
$$h - d = \frac{1}{2}g\left(\frac{v}{a}\right)^2 \implies v > a\sqrt{\frac{g}{2(h-d)}}.$$

5. A runner is on a track. When the gun is fired to start the race, he begins running with his speed increasing over time. His speed is proportional to the frequency of his legs that propel him forward. Which of the graphs below show the amount force exhibited by his legs as he runs faster?

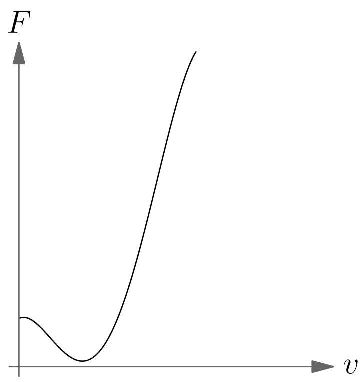
(A)



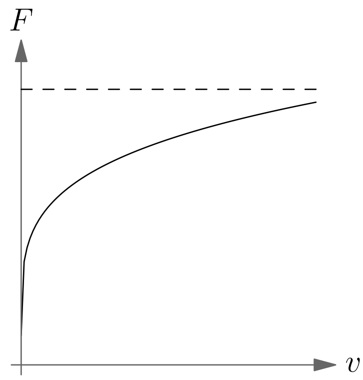
(B)



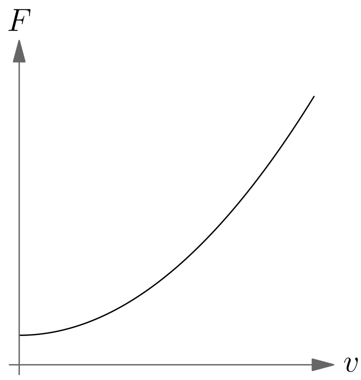
(C) ← **CORRECT**



(D)



(E)



Solution: An important part of this problem is that is is said that the runners speed is proportional to the frequency of the legs that propel him. That means that at some point in his motion, the runner will reach a resonant frequency. At resonant frequency, the runner will have to exert the least effort to propel himself further. One can also see this when you are running, there is a certain pace at which you feel the "lightest" on your feet. So, the graph will initially dip before the runner goes to the resonant frequency and then once again rise because it obviously takes more a lot of effort and force with more and more velocity. This is why humans can never go faster than a certain speed because of the amount of force it takes for them to move.

6. There are two ramps that have an angle of θ_1 and θ_2 respectively. The first ramp has a coefficient of friction that is almost infinite while the second has a coefficient of friction of 0. The ramps are made

such that they have the same horizontal length. Two cylinders of equal uniform density and size are placed on both ramps and the time to reach the bottom is measured to be the same. Under these conditions, find the angle θ_2 that minimizes the time for the cylinders to descend the ramps.

(A) $19°$
(B) $21°$ ← **CORRECT**
(C) $30°$
(D) $45°$
(E) Due to the infinite coefficient of friction in the first ramp, the first cylinder will never descend to the bottom.

Solution: First we must find a relationship between θ_1 and θ_2 such that the cylinders reach the bottom simultaneously. To do this, we consider the accelerations of each down the incline and the distance that each must cover. Using this, we can solve for time. For the cylinder on the incline with friction,

$$a_1 = \frac{2g\sin\theta_1}{3},$$

$$d_1 = x\sec\theta_1,$$

and for the cylinder on the incline without friction,

$$a_2 = g\sin\theta_2$$

$$d_2 = x\sec\theta_2$$

where x is the horizontal distance. Using the big-five formula, $\Delta s = \frac{1}{2}at^2 + v_0 t$ with $v_0 = 0$, we can solve for time and simplify.

$$t_1 = \sqrt{\frac{6d}{g\sin 2\theta_1}}$$

$$t_2 = \sqrt{\frac{4d}{g\sin 2\theta_2}}$$

Setting these two equations equal and rearranging yields,

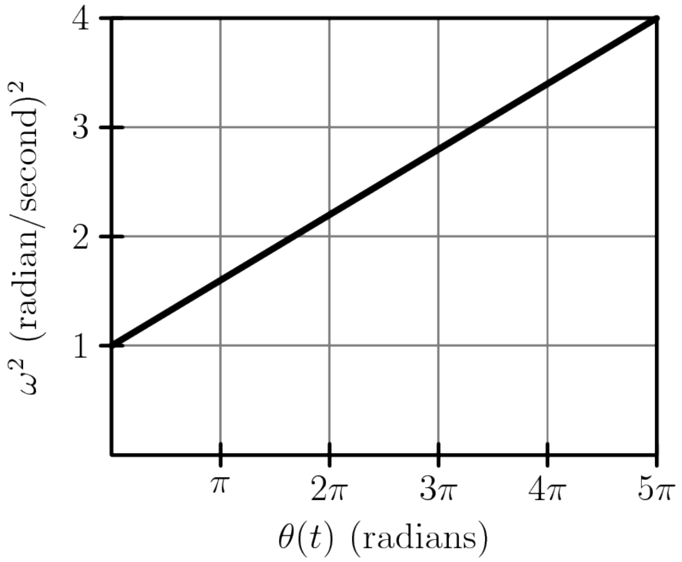
$$3\sin 2\theta_2 = 2\sin 2\theta_1.$$

Since the time to go down the hill is inversely proportional to $\sin 2\theta_2$, we wish to maximize $\sin 2\theta_2$. The maximum value it can attain given constraints is $\frac{2}{3}$. Solving for theta yields,

$$\theta_2 = \frac{1}{2}\sin^{-1}\frac{2}{3}$$

$$\boxed{\theta_2 = 21°}$$

7. A gear with moment of inertia 50 kg m^2 is rotating under the influence of an external agent as shown below. A $\omega^2 - \theta(t)$ diagram of the gear's rotation is plotted as shown below. What is the value of the external torque that acts on the gear?



(A) 2.7 N · m
(B) 4.8 N · m ← **CORRECT**
(C) 7.3 N · m
(D) 9.5 N · m
(E) 19.1 N · m

Solution: Note that since torque is constant, it can be written that

$$\tau \sum \Delta\theta = \sum \Delta E \implies \tau = \frac{I}{2\theta}(\omega_f^2 - \omega_i^2).$$

Substituting values gives us approximately 4.8 N · m.

One can also use calculus by primarily using the fact that

$$\frac{\mathrm{d}(\omega^2)}{\mathrm{d}\theta} = 2\omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = 2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = 2\frac{\mathrm{d}\omega}{\mathrm{d}t}.$$

Since $\tau = I\frac{\mathrm{d}\omega}{\mathrm{d}t}$, it can be written that $\tau = \frac{1}{2}I\frac{\mathrm{d}(\omega^2)}{\mathrm{d}\theta} = \frac{15}{\pi}\mathrm{N}\cdot\mathrm{m}$.

8. A rope is hung between two poles such that it sags down when at rest. The bottom-most point of the rope is grabbed and pulled down such that the sides of the rope become completely straight and make a triangular shape with the poles. How did the center of mass of the rope change?

(A) It moved to a lower position.
(B) It moved to a lower position if the curvature of the rope is initially small, and a higher position if the curvature of the rope was big.
(C) It remains in the same position.
(D) It moved to a higher position. ← **CORRECT**
(E) It moved to a higher position if the curvature of the rope is initially small, and a lower position if the curvature of the rope was big.

Solution: Note that the rope is in a position that assumes the least potential energy. To move the rope in any way, you must apply some force which does positive work and therefore increases the potential energy of the rope. Thus, the center of mass will elevate (as $U = mgh$ where h is the height of the center of mass) as a result of added energy.

9. A small box A is placed on a floor with coefficient of friction 0.2. A is located a distance of 1 m from an identical box B of the same dimensions. A is then released with a velocity of 3 m/s towards B. Assuming the collision to be entirely inelastic, what is the total distance the box travels from its initial position after it stops?

(A) 1.31 m ← **CORRECT**
(B) 1.45 m
(C) 1.63 m
(D) 1.88 m
(E) 1.96 m

Solution: Let us first conserve energy to find the speed of the block A, u before it collides with block B:

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 - \mu mgx \implies u = \sqrt{v^2 - 2\mu gx}.$$

where $x = 1$ m. One can now write that

$$mu = 2mu' \implies u' = \frac{1}{2}\sqrt{v^2 - 2\mu gx}.$$

Once again conserving energy yields

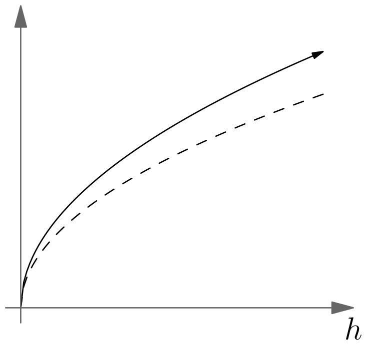
$$\frac{1}{2}mu'^2 = \mu(2m)gy \implies y = \frac{1}{4\mu g}(v^2 - 2\mu gx).$$

The total distance travelled is then $x + y = 1.31$ m.

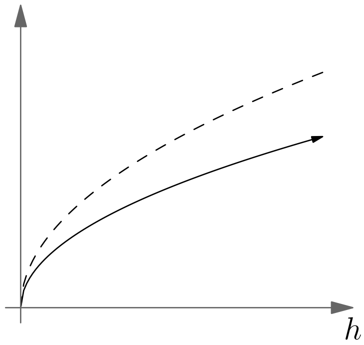
The following information is relevant to problems 10, 11, and 12.

A *snow* cylinder with an initial radius R rolls without slipping down a tall hill with constant slope. As it rolls, snow sticks onto the cylinder which makes its radius slowly increase. The amount of gathered snow is proportional to the distance the snow cylinder has travelled. Consider the following graphs which represent arbitrary units on the y-axis which will be specified according to the problem. (Note: For graph III, the arrow lies directly on top of the dotted line).

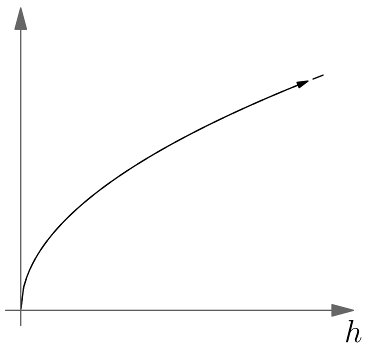
(I)



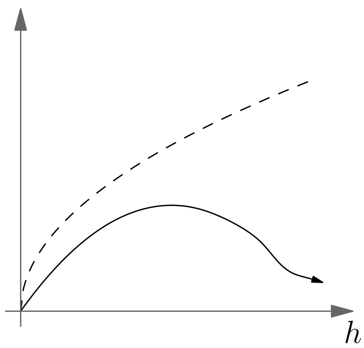
(II)



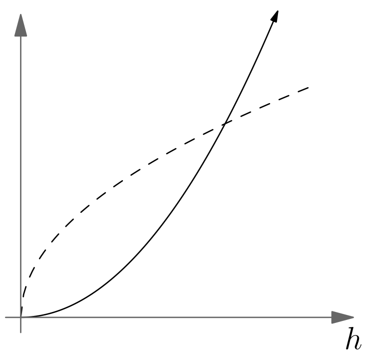
(III)



(IV)



(V)



10. If the dotted line represents the speed of the cylinder if the snow was not sticky, which of the following graphs represents the speed of the cylinder if the snow sticks to the cylinder, as described above?

(A) Graph I
(B) Graph II
(C) Graph III ← **CORRECT**
(D) Graph IV
(E) Graph V

Solution: Since the object is rolling without slipping, we have the relationship $v = \omega r$. Applying energy conservation allows us to write that

$$mgh = \frac{1}{2}mv^2 + I\omega^2 \implies v \propto \sqrt{h}.$$

From this analysis, it is evident that the changing radius of the cylinder does not actually effect the velocity it rolls down with.

11. If the dotted line represents the angular speed of the cylinder if the snow was not sticky, which of the following graphs represents the angular speed of the cylinder if the snow sticks to the cylinder, as described above?

(A) Graph I
(B) Graph II
(C) Graph III
(D) Graph IV ← **CORRECT**
(E) Graph V

Solution: Since $\omega = v/r$, the angular speed of the snow cylinder will asymptote to zero when $r \to \infty$. Contrarily, for the regular cylinder $\omega = v/r$ will only have the speed changing which is given to be $v \propto \sqrt{h}$. Hence, graph IV is the only graph that represents this.

12. Instead of the cylinder, we roll a large snowball down the hill. Assume the ball travels in a straight line and snow sticks to only the parts of the surface that makes contact with the ground. If the dotted line represents the speed of the ball if the snow was not sticky, which of the following graphs represents the speed of the ball if snow sticks to it?

(A) Graph I
(B) Graph II ← **CORRECT**
(C) Graph III
(D) Graph IV
(E) Graph V

Solution: Since snow only sticks to the parts the snowball makes contact with, the coefficient of the moment of inertia $\beta > 2/5$ since the snowball will eventually look like a sphere connected with a washer ring adding to more inertial resistance. Hence, we know that from problem 10, the energy conservation equations will give us

$$v_{\text{snow}} = \sqrt{\frac{gh}{1/2 + \beta}}, \text{ and } v_{\text{normal}} = \sqrt{\frac{gh}{1/2 + 2/5}}.$$

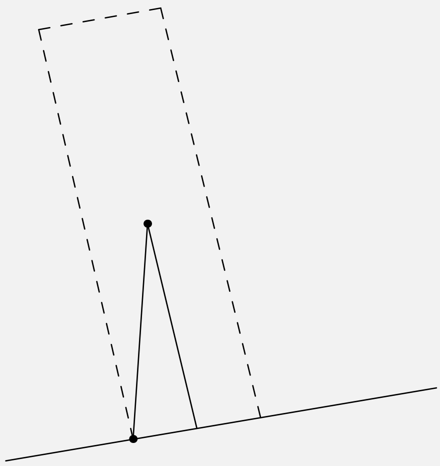
From here, $v_{\text{normal}} > v_{\text{snow}}$ but both have the same graph shape. This gives graph II.

13. Books are stacked on each other atop a flat desk that is tilted at an angle of 10° with respect to the horizontal. The coefficient of friction between every surface is $\mu = 0.2$ and a book can be thought of as a rectangular prism of length and breadth 10 cm, and thickness of 3 cm. The first book is placed on the desk with an arbitrary orientation with respect to the desk. All subsequent books are placed exactly over the previous book (the resulting stack should be a rectangular prism). What is the maximum number of books we can stably stack on the desk (i.e. the stack does not slide or tip over)?

(A) 22
(B) 23

(C) 24
(D) 25
(E) 26 ← **CORRECT**

Solution: As $\mu > \tan\theta$ Books will never slide.
Let n be the maximum number of books. Since the thickness of each book is 3 cm, the distance from the center of mass of the entire book stack to the ramp will be $3n/2$. Similarly, we can maximize the total horizontal breadth of the way the books are stacked by placing them such that they make a 45° angle. That means that the total diagonal length is $10\sqrt{2}$.



We then construct a right triangle as shown below. By using trigonometry, we can see that if θ is the lower angle of the right triangle, then

$$\theta + 10^\circ < 90^\circ \implies \arctan\left(\frac{3n/2}{5\sqrt{2}}\right) < 80^\circ \implies n < 27.$$

14. A sink is filled with water of density 1000 kg/m^3. A small wooden rectangular block of uniform density 600 kg/m^3 is then placed on top of this water layer. Soon after, a layer of dish soap with density 250 kg/m^3 is poured into the sink until it completely covers the block. Find the ratio of the thickness of the soap layer to the thickness of the block.

(A) 1/5
(B) 7/15
(C) 8/15 ← **CORRECT**
(D) 4/5
(E) 7/8

Solution: Let the density of water be ρ_w, then density of the block ρ_b, and the density of soap ρ_s. We can now balance forces noting that the buoyant forces are directed upwards and the weight forces are directed downward. Consider a cross section with the same breadth as that of the soap block, total vertical length L, and the block with width x. Balancing forces yields $\rho_b g x = \rho_w g(L - x) + \rho_s g x$. Solving for x now yields 8/15.

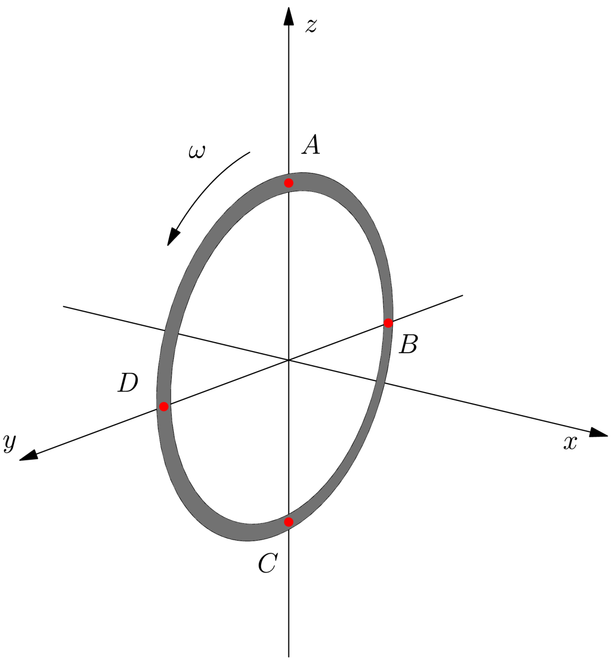
15. A planet is able to expand to a radius 2 times its initial radius. Approximately by what factor has the pressure in the atmosphere of the planet changed? Assume that the mass of the planet remains the same before and after the expansion, and no air escapes its atmosphere. Take the universal gravitational constant to be $G = 6.7 \times 10^{-11}\ \mathrm{N\cdot m^2/kg^2}$.

(A) 1/16 ← **CORRECT**
(B) 1/4
(C) 2

(D) 4
(E) 8

Solution: The only constants that will be involved in the pressure of the planet will be the universal gravitational constant G, the mass M of the planet, and the radius R of the planet. Hence, through simple dimensional analysis we find that $P = \alpha\frac{GM^2}{R^4}$, where α is a numerical constant. Comparing between before and after the radius doubles, one can then find that the pressure is 1/16 smaller.

16. A ring is rotating in the $y-z$ plane as shown in the figure below. The system is isolated and gravity may be neglected. Jack applies a constant force $F\hat{x}$ at some position on the ring which changes the orientation to move along the $x-z$ plane. At which of the following positions was the impulse applied?



(A) A ← **CORRECT**
(B) B
(C) C ← **CORRECT**
(D) D
(E) The position of the ring will always change irrespective of the position the impulse is applied.

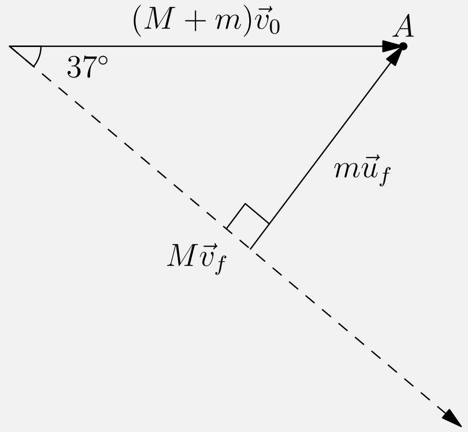
Solution: This setup essentially follows the idea of gyroscopic precession. If the torque is aligned with the angular momentum of the ring, then the angular velocity simply changes. However, if the torque is not aligned with the angular momentum, then a change in the angular momentum must occur inducing a precession and change in orientation. By this, that means that both (A) and (C) are correct.

17. A spaceship has a total mass of 500 kg and is moving at a constant speed of 2 m/s. The passengers throw 50 kg of cargo out of the spaceship all at once at a constant speed, causing the spaceship to move at a 37° angle with respect to its initial path. Find the minimum speed of the cargo in the frame of the spaceship in order to achieve this course correction. The ship and the cargo can be thought of as point particles.

(A) 8 m/s
(B) 12 m/s ← **CORRECT**

(C) 15 m/s
(D) 18 m/s
(E) 21 m/s

Solution: We can construct a momentum diagram as shown below:

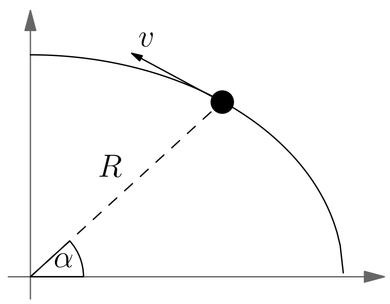


As seen, the shortest distance between the point and the vector $M\vec{v}_f$ can be traced by pointing the momentum vector $m\vec{u}_f$ of the ejected cargo at point A. This then creates a right triangle which allows us to write that

$$mu_f = (M+m)v_0 \sin(37)$$

which yields answer choice B.

18. A wire is placed in a two dimensional coordinate plane which follows the polar equation $r(\alpha) = \beta e^{\gamma\alpha}$ where r is the distance from the origin to a point on the wire and α is the angle from the origin to the same point. A small bead of mass m is then put onto this wire and the whole system is moved to a vacuum where the effects of gravity are negligible. The bead is then pushed to move on this wire. If the speed of the bead at an arbitrary angle α with respect to the origin and the x-axis is v, find the normal force experienced by the bead from the wire. The distance from the origin and the bead at this moment of time is given to be R.



(A) $\dfrac{mv^2}{R\sqrt{1+\gamma^2}}$ ← **CORRECT**
(B) $\dfrac{mv^2}{R\sqrt{\beta^2+\gamma^2}}$
(C) $\dfrac{\gamma mv^2}{R\sqrt{1+\gamma^2}}$
(D) $\dfrac{\beta mv^2}{R\sqrt{1+\beta^2}}$
(E) $\dfrac{mv^2}{R\sqrt{1+\beta^2}}$

Solution: Note that in the limit of $\gamma \to 0$, the polar equation will turn into $r(\alpha) = \beta$ or a circle with a radius β. The centripetal force mv^2/R must then balance the normal force N in this limit which only gives answer choice A.

Alternatively, if one knows calculus, they can use the following formula for the radius of curvature in polar coordinates $\mathcal{R}$ given to be

$$\mathcal{R} = \frac{(r^2 + r_\alpha^2)^{3/2}}{|r^2 + 2r_\alpha^2 - r r_{\alpha\alpha}|}$$

where r_α and $r_{\alpha\alpha}$ are the first and second derivative respectively. Substituting shows us

$$\begin{aligned}\mathcal{R} &= \frac{(\beta^2 e^{2\gamma\alpha} + \beta^2\gamma^2 e^{2\gamma\alpha})^{3/2}}{|\beta^2 e^{2\gamma\alpha} + 2\beta^2\gamma^2 e^{2\gamma\alpha} - \beta^2\gamma^2 e^{2\gamma\alpha}|} \\ &= \frac{[\beta^2 e^{2\gamma\alpha}(1+\gamma^2)]^{3/2}}{\beta^2 e^{2\gamma\alpha}(1+\gamma^2)} \\ &= \beta e^{\gamma\alpha}\sqrt{1+\gamma^2} \\ &= R\sqrt{1+\gamma^2}\end{aligned}$$

Therefore, the ball can be thought of as moving in an imaginary circle of radius $\mathcal{R}$ and the centripetal force will be equal to the normal force that opposes it (as there is no gravity):

$$N = \frac{mv^2}{\mathcal{R}} = \frac{mv^2}{R\sqrt{1+\gamma^2}}.$$

19. Two identical masses are attached to the same spring and are placed in a vacuum in space. They are made to oscillate at an oscillation rate ω. One of the masses loses its mass at a slow constant rate while the other mass increases its mass at the same constant rate (so the total mass always remains the same). This process happens for some time $\tau < t$ where t is the time for one of the masses to become zero. Which of the following statements is always true about the new oscillation rate Ω?

(A) $\Omega < \omega$
(B) $\Omega > \omega$ for short time periods; $\Omega < \omega$ for long time periods.
(C) $\Omega = \omega$
(D) $\Omega < \omega$ for short time periods; $\Omega > \omega$ for long time periods.
(E) $\Omega > \omega$ $\leftarrow$ **CORRECT**

Solution: Let the initial masses be labeled m. It is known that the oscillation rate is

$$\omega = \sqrt{\frac{\kappa}{\mu}},$$

where κ is the spring constant and $\mu \equiv \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of a system formed by two masses m_1 and m_2. This fact is proved at the end of this solution. At time t, let the two masses have the new values $m \pm \Delta m$ (they are modified at the same rate, so their variations are equal in absolute value). The expressions of the two oscillation rates are

$$\omega = \sqrt{\frac{2m\kappa}{m^2}} \text{ and } \Omega = \sqrt{\frac{2m\kappa}{(m-\Delta m)(m+\Delta m)}} = \sqrt{\frac{2m\kappa}{m^2 - (\Delta m)^2}}.$$

The denominator of the second fraction is smaller than the denominator of the first fraction, since $(\Delta m)^2$ is necessarily positive, so we conclude that $\Omega > \omega$.
Proof: The derivation is going to be carried out in the COM frame of reference (as there are no external forces, this frame is inertial). Let the displacements of the two masses be x_1 and x_2. Then, the spring is elongated by $x_2 - x_1$ and thus exerts a force $F = \kappa(x_2 - x_1)$ on either

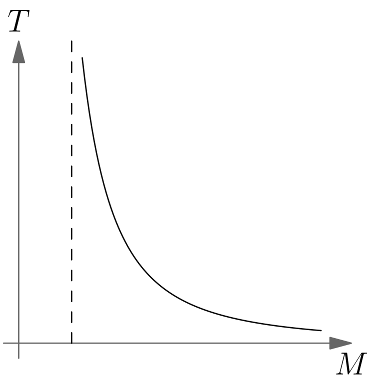
of the masses. Due to the fact that the COM is at rest, $m_1 x_1 + m_2 x_2 = 0$, which means that $F = -\kappa x_1 (m_1/m_2 + 1)$. Newton's 2^{nd} law reads $F = m_1 a_1$, so

$$a_1 = -\frac{\kappa}{m_1}\left(\frac{m_1}{m_2} + 1\right) x_1,$$

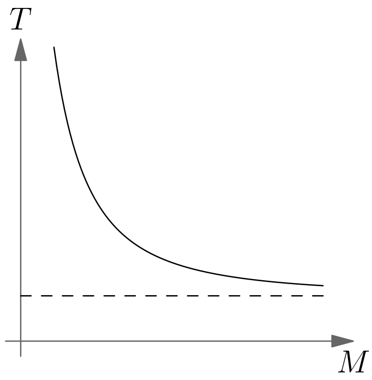
which is the equation of motion for an oscillator with $\omega^2 = \kappa\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \kappa\frac{m_1 + m_2}{m_1 m_2} \equiv \frac{\kappa}{\mu}$.

20. A planet is revolving around a star which has a mass significantly greater than that of the planet. Everytime the planet completes a period, the star has its mass slightly reduced. This process continues repeatedly. Which of the following graphs best represent the mass dependence of the star M to the period of the planet T. *Note:* The x-axis corresponds to a mass that is still much greater than the planet.

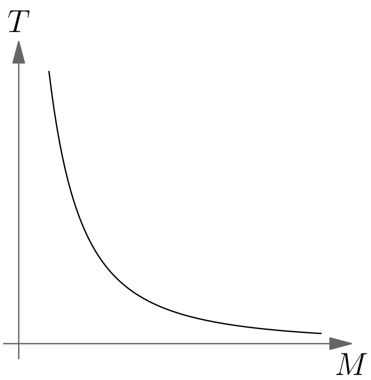
(A) ← **CORRECT**



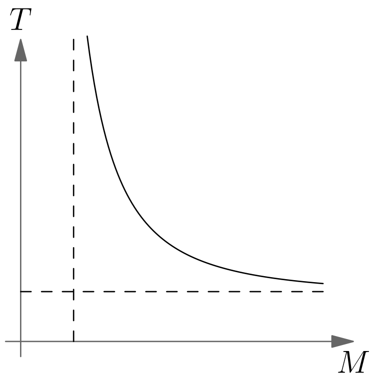
(B)



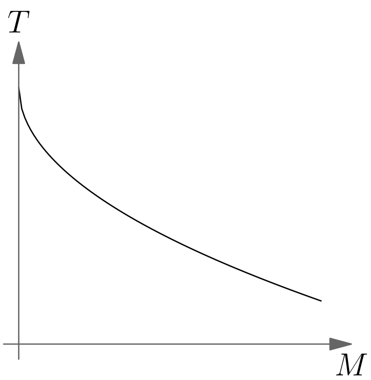
(C)



(D)



(E)



Solution: Let the fixed point be A where the planets comes and star losses his mass. Velocity there will not be changed due to removal of the mass. Now when the planet again come to point A its velocity will be same as before. So we can write equation of total energy at that point. Which is

$$-\frac{GMm}{2a} = -\frac{GMm}{r} + \frac{1}{2}mv^2$$

here a is semimajor axis and r is the distance of point A to the star. Also we can write equation of time period as

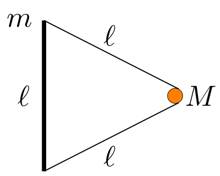
$$T = 2\pi\sqrt{\frac{a^3}{GM}}$$

substituting value of a in the first equation and simplifying it we get

$$T = \frac{M}{(cM - d)^{\frac{3}{2}}}$$

here c and d are any arbitrary constant such that cM-d>0. From this equation we can clearly see that it will give graph A

21. On a horizontal, frictionless table lies a uniform rod of length $\ell = 1$ m and mass m. A light, elastic thread of relaxed length $\ell_0 = \ell\sqrt{3}$ has its ends attached to the ends of the rod. A small bead of mass $M = m/2$ is embedded on the thread and fixed at its midpoint (the bead is not stuck the cord). The thread is pulled by its midpoint away from the rod until each of its two portions have the same length as the rod, as shown below. Find the distance covered by the bead while it accelerates, after the system is released.



Top view of the system just before it is released.

(A) 0.866 m
(B) 0.106 m ← **CORRECT**
(C) 0.081 m
(D) 0.577 m
(E) 0.244 m

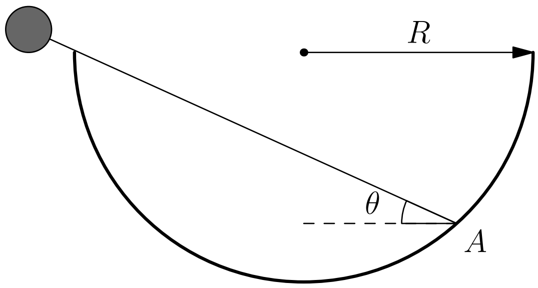
Solution: Initially, the COM is at a distance $d_1 = \dfrac{m\ell\sqrt{3}}{2(m+M)}$ away from M. When the accelerated motion stops, the elastic force in the thread cancels, which means that the thread reaches its natural length. At this moment, the COM is at a distance $d_2 = \dfrac{m\sqrt{\ell_0^2/4 - \ell^2/4}}{m+M} = \dfrac{m\sqrt{\ell_0^2 - \ell^2}}{2(m+M)}$ away from M. The table is frictionless, meaning that there are no external forces acting on the system. Therefore, the COM is stationary with respect to the table, meaning that the distance covered by the bead in accelerated motion is simply

$$d = d_1 - d_2 = \frac{m\ell\sqrt{3}}{2(m+M)} - \frac{m\sqrt{\ell_0^2 - \ell^2}}{2(m+M)} = \frac{m\ell}{2(m+M)}\left(\sqrt{3} - \sqrt{\frac{\ell_0^2}{\ell^2} - 1}\right)$$

which gives $d = \dfrac{\sqrt{3} - \sqrt{2}}{3}$ m ≈ 0.106 m.

The following information is relevant to problems 22 and 23.

A lollipop is put into a semi-spherical hole of radius R with the inside of its surface having a coefficient of friction of μ. The lollipop consists of a large spherical mass M connected to a uniform rod of mass m and length $2R$. The lollipop makes an angle θ with respect to the horizontal and the hole as shown in the picture and the hole is fixed.



22. What is the minimum value of the absolute coefficient of friction $|\mu|$ of the inner surface of the hole such that the lollipop stays in place?

(A) $\dfrac{M\cos\theta}{M+m}$

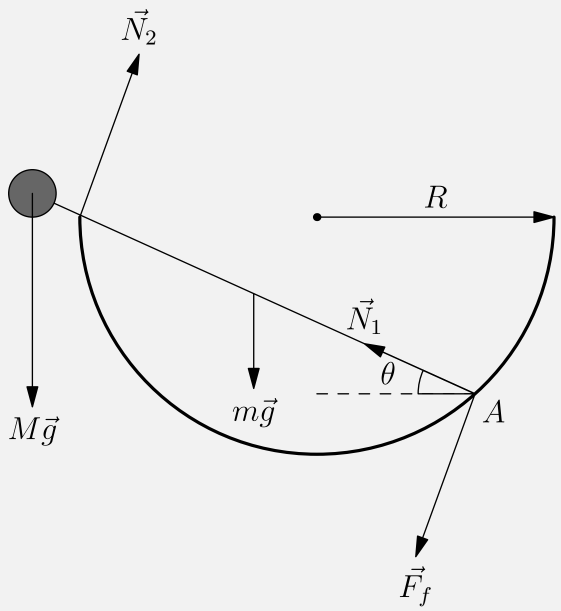
(B) $\dfrac{3M}{2(M+m)\sin\theta} + \tan\theta$

(C) $\dfrac{2(M+m)}{5(M+m)\cos\theta} - \tan\theta$

(D) $\dfrac{2M+m}{2(M+m)\sin\theta} - \cot\theta$ ← **CORRECT**

(E) $\dfrac{3M+m}{(M+m)\sin\theta} - \cot\theta$

Solution: The forces acting on the lollipop are shown in the diagram below.



The lollipop stays in place as long as it remains in translational and rotational equilibrium. Equilibrium condition along a direction parallel to the rod:

$$N_1 = Mg\sin\theta + mg\sin\theta.$$

Notice geometrically that the overhang of the lollipop is $2R - 2R\cos\theta$. Torque balance condition about the upper point of contact with the hemisphere:

$$\vec{\tau}_{M\vec{g}} + \vec{\tau}_{m\vec{g}} + \vec{\tau}_{\vec{F}_f} + \cancel{\vec{\tau}_{\vec{N}_1}} + \cancel{\vec{\tau}_{\vec{N}_2}} = \vec{0},$$

$$2MgR(1-\cos\theta)\cos\theta - mgR(2\cos\theta - 1)\cos\theta - 2F_f R\cos\theta = 0.$$

After straightforward algebraic manipulation, the expression of F_f turns out to be

$$F_f = \frac{2M(1-\cos\theta) - m(2\cos\theta - 1)}{2}g,$$

which, combined with the fact that $F_f = \mu N_1$, yields

$$\mu = \frac{2M(1-\cos\theta) - m(2\cos\theta - 1)}{2(M+m)\sin\theta} = \frac{2M+m}{2(M+m)\sin\theta} - \cot\theta.$$

23. At point A, the end of the lollipop is moved by an external source at a constant velocity modulus $|v|$ *along the surface* of the hole. Which of the following expressions best represents the kinetic energy of the mass M as a function of θ? Neglect any frictional effects.

(A) $\frac{1}{2}Mv^2\sin^2\theta$

(B) $Mv^2 \cdot \frac{1-\cos\theta}{\cos^2\theta}$

(C) $2Mv^2\sin^2(\theta/2)$ ← **CORRECT**

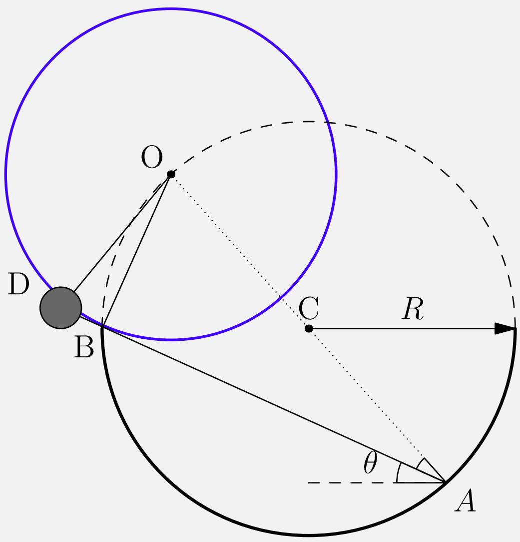
(D) $\frac{1}{2}Mv^2$

(E) $Mv^2(1-\cos\theta)$ ← **CORRECT**

Solution: Let point D be the position of the mass M and point B be a point on the end of the hole. Let us consruct a circle with center at point C as shown in the figure below.

Claim. Point O is centered at the instantaneous axis of rotation of point B.

Proof. As the end of the rod moves with an angular velocity $\omega = v \times 2R$ on the bottom half of the semicircle, the same velocity can be mirrored diametrically opposite to A. Note that $\angle ABO = 90°$ which will always be achieved in a configuration as shown below. The velocity of point B will then be orthogonal to a circle that describes a radius of OB centered at O which means that circle O is the instantaneous center of rotation of point B.



Hence, we can now write

$$v = \omega \cdot AO = \frac{v}{2R} \cdot \sqrt{(DB)^2 + (BO)^2} = 2v\sin(\theta/2).$$

The kinetic energy is then

$$T = \frac{1}{2}Mv^2 = 2Mv^2\sin^2(\theta/2).$$

24. A solid ball of mass m and radius R lies at rest on a horizontal table in Earth's gravitational field. At $t = 0$, a horizontal force F starts pushing the ball, from a point on its surface situated at height h above the ground, such that the ball is engaged in solely translational motion. After $t = t_1$, the force is no longer applied. Find the number of full rotations will the ball undergo until it starts rolling without slipping.

(A) $\dfrac{3Ft_1^2h^2}{25mR^2(R-h)}$

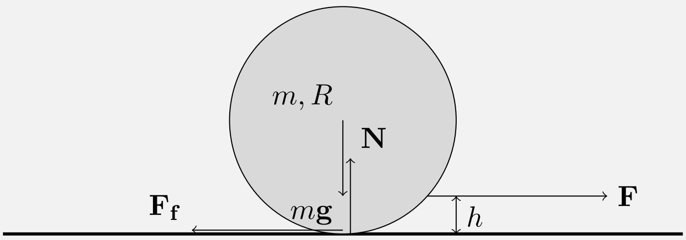
(B) $\dfrac{5Ft_1^2h^2}{49mR^2(R-h)}$

(C) $\dfrac{3Ft_1^2h^2}{50mR^2\pi(R-h)}$

(D) $\dfrac{5Ft_1^2h^2}{98mR^2\pi(R-h)}$ ← **CORRECT**

(E) The ball will never begin to roll without slipping.

Solution: The forces acting on the ball are indicated in the diagram.



The fact that the ball is engaged in solely translational motion implies that there is a friction force whose torque precisely cancels the torque produced by F,

$$F_f R = F(R-h)$$

Newton's 2nd law projected on a horizontal axis reads

$$F - F_f = ma \implies F - F\frac{R-h}{R} = ma \implies a = \frac{Fh}{mR}$$

which means that the ball accelerates steadily. As such, the speed of the ball at t_1 is given by

$$v_1 = at_1 = \frac{Fht_1}{mR}$$

After $t = t_1$, if F stops acting on system, the friction force will produce a torque which will steadily decrease v and increase the angular velocity of the ball, as follows

$$a' = -\frac{F_f}{m} \implies v(t) = v_1 - F\frac{R-h}{mR}(t-t_1)$$

$$F_f R = I\varepsilon \implies \varepsilon = F\frac{R-h}{I} = 5F\frac{R-h}{2mR^2}$$

Since the angular acceleration is constant, the angular velocity and angular displacement can be expressed as

$$\omega(t) = \varepsilon(t-t_1) = 5F\frac{R-h}{2mR^2}(t-t_1)$$

$$\Delta\theta(t) = \frac{1}{2}\varepsilon(t-t_1)^2 = 5F\frac{R-h}{4mR^2}(t-t_1)^2$$

The ball starts rolling without slipping when $v = \omega R$, meaning that

$$v_1 - F\frac{R-h}{mR}(t-t_1) = 5F\frac{R-h}{2mR}(t-t_1) \implies t - t_1 = \frac{2mRv_1}{7F(R-h)}$$

The number of full rotations will thus be

$$N = \frac{\Delta\theta}{2\pi} = \frac{5mv_1^2}{98\pi F(R-h)} = \frac{5Ft_1^2h^2}{98mR^2\pi(R-h)}$$

25. A city planner decided to conduct a mini study to see how popular a certain road was. He counted the number of cars that passed through the road on two random days. On the first day, he counted 500 cars in 30 min. On the next day, he counted 200 cars in 10 min. The city planner decided to average the rate of car flow on both days. What is the uncertainty he should report?

(A) ± 1 cars/min $\leftarrow$ **CORRECT**
(B) ± 3 cars/min
(C) ± 5 cars/min
(D) ± 7 cars/min
(E) ± 10 cars/min

Solution: let us say the uncertainty for each random variable was δx. If you have N such instances, then the uncertainty is:

$$\begin{aligned}\sigma &= \sqrt{\delta x^2 + \delta x^2 + \cdots + \delta x^2} \\ &= \sqrt{N}\delta x\end{aligned}$$

On the first day, the uncertainty per minute was $\sqrt{500}/30 = 0.75$ cars/min. On the second day, the uncertainty per minute was $\sqrt{200}10 = 1.41$ cars/min. Since these were two random days, the overall uncertainty would be

$$\delta N = \sqrt{0.75^2 + 1.41^2} = 1.60 \text{ cars/min.}$$

However, since there cannot be a fraction of the number of cars, we must round down such that the number of significant figures stays the same. In other words, the uncertainity becomes 1 cars/min.