2020 Taiwan International/Asian Physics Olympiad (IPhO/APhO) Selection Programme

Team Selection Tests (Theoretical Exams)

Translated by: **Wayne Yan**

Edited by: **Kushal Thaman**

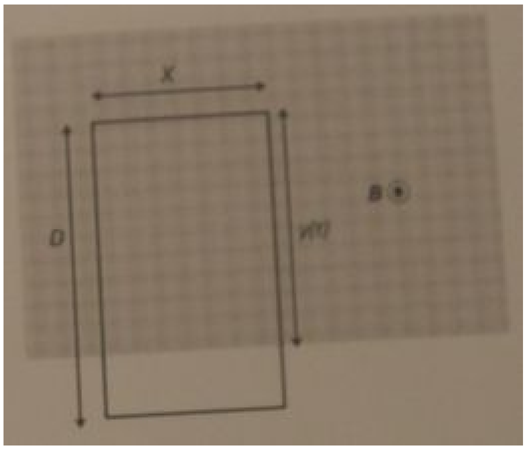
Theoretical Test 1

Total Marks: 100

Time: 4 hours

1. The electromagnetic induction and motion of a coil

(*Score: 20 points*)



As the figure shows, a rectangular coil of length D, width X, mass m, is put in a uniform magnetic field B, and the magnetic field is perpendicular to the surface of the coil. At time $t = 0$, the bottom side of the coil is situated at the edge of the uniform magnetic field. Let $y(t)$ denote the relative position between the top side of the coil and the edge of the magnetic field. The above initial condition is then $y(0) = D$. Let the positive direction of current be counter-clockwise, and upward is the positive y-direction.

Part A

The coil is released at $t = 0$, and suppose that D is big enough that the coil reaches terminal velocity v_t before exiting the field. The inductance of the coil is negligible, and its resistance is R. Answer the following questions with parameters X, D, B, m, R, t and acceleration due to gravity g.

(i) Find the magnitude of v_t.

(ii) Find the relation between the speed of the coil and time before the coil exits the field.

(iii) Find the relation between the power used by the coil and time before the coil exits the field.

(iv) Find the total energy the coil uses from $t = 0$ till the coil exits the field.

Part B

Let the coil be a superconductor, but its inductance L is no longer negligible. Suppose D is big enough that the coil never falls out. Answer the following questions with parameters X, D, B, m, L, t and gravitational acceleration g.

(v) Prove the coil performs SHM.

(vi) Find $y(t)$.

(vii) Find the maximum kinetic energy of the coil.

(viii) Find the maximum energy stored in the magnetic field of the coil, and when it happens.

2. A slowly shortened pendulum (*Score: 20 points*)

A pendulum is hung below the ceiling, with a point mass m on the other end, the string holding them is massless, and gravitational acceleration is g. now we shorten the string extremely slowly, so $\frac{L}{\dot{L}}$ is much greater than the period of the pendulum.

(A) When the string's length is L, and the amplitude is θ_0. What is the average tension of the string?

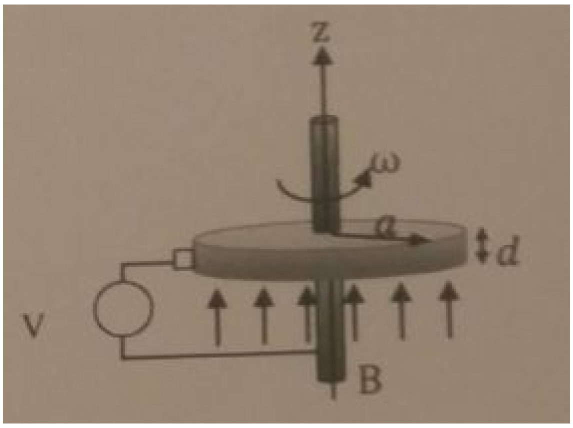
(B) When the string's length is L, and the amplitude is θ_0, the length is shortened by $\Delta L < 0$, the amplitude changed by $\Delta\theta_0$. What is the change of the mechanical energy of the pendulum?

(C) Using the above results, find the amplitude when the length of the string is one half of the original length.

3. The charge distribution of a rotating metal disk in a static magnetic field and the mechanism of the earth's magnetic field. (*Score: 20 points*)

Part A

As the figure shows, in an external magnetic field B along z-axis in the region $r \le a$, a neutral metal disk of radius a is spinning along an metallic axle with angular speed ω, the radius of the axle can be ignored, and the conductivity of the metal is σ, when the current and charge distribution is in a stable state, we find the potential difference between the edge of the plate and the axle is V, answer the following questions. The permittivity of vacuum is ε_0.

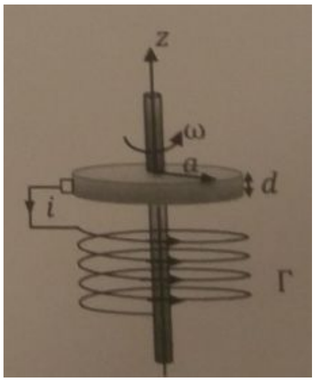


(i) When the current and charge distribution is in a stable state, let the electric field of $(\vec{r}, z)$ is $\vec{E}$, and the magnetic field is $\vec{B}$, deduce the relation between the electric field and the magnetic field, and then find V.

(ii) Ignore boundary effects, and view the disk as a long cylinder, consider the electric field flux between radius r and $r + dr$. Find the charge distribution in the stable state.

(iii) Consider the current crossing radius r and $r + dr$, estimate $\frac{d\rho}{dt}$ at $(\vec{r}, z)$, where ρ is the charge density. Estimate how long it takes for the current and charge distribution to reach equilibrium, from the moment it starts spinning.

<u>Part B</u>



As the figure shows, with certain configuration, we can , without an external field, feed the current produced by the spinning disk to the coil Γ to create a magnetic field, finally getting a static magnetic field. This is also a simple model of the Earth's magnetic field. The self inductance of the circuit is L (mostly from Γ) and the resistance is R. For the coil, there are n turns per unit length, the length of the coil is l and its radius is a. The moment of inertia of the disk and the axle is I, and the rotation damping coefficient is γ (the resisting torque is $-\gamma\omega$). Answer the following questions.

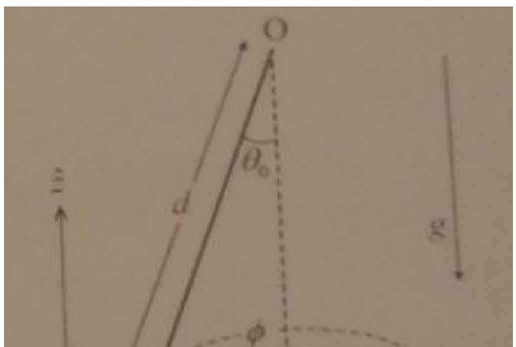
(i) Suppose the distance between the coil and the disk is negligible, and the field produced by the coil is uniform, equal to the field produced at the center, find the self inductance of the coil L, and the mutual inductance between the coil and the disk. when the angular speed of spinning is ω, and current in the circuit is i, find the relation of L, M, R, ω and i.

(ii) When the current of the coil is i, find the torque on the disk. (take the upward torque as positive)

(iii) From (i) and (ii) we can find the equation between i and ω, suppose a constant torque τ_0 is acting on the axle to make it rotate, and $\tau_0 > 2\pi\gamma\frac{R}{M}$, $\omega(0) = \omega_0 > \frac{2\pi R}{M}$. If $i(0) = 0$, what is the equilibrium angular speed ω_e and the current i_e? If $i(0) = 0.01$, what is ω_e, i_e in that case?

(iv) With the condition $\tau_0 > 2\pi\gamma\frac{R}{M}$, plot $\omega - i$ graph with different initial conditions of $i(0), \omega(0)$.
Note: Consider $i(0), \omega(0)$ is $i(t), \omega(t)$ at time $t = 0$.

4. Conical Pendulum (*Score: 20 points*)



Consider a pendulum system with a point mass m, and a string with length d, as the figure shows. Gravitational acceleration is g, ignore the mass of the string and the spinning of the Earth. The initial condition of the system is $\theta = \theta_0$, $\dot{\theta} = \frac{d\theta}{dt} = 0$, and the velocity of the mass is $-v_0 \hat{y}$ (when the pendulum is on the x-z plane).

(A) Find the angular momentum of the system L_z relative to the pivot O.

(B) Find the total energy of the system E_0, and express your answer in the terms of θ, $\dot{\theta}$, $\dot{\varphi}$.

(C) If the minimum value of θ in the motion is θ_{min}, find v_0.

(D) Prove that the form of $\frac{d\varphi}{d\theta}$ is given by the differential equation

$$\frac{d\varphi}{d\theta} = \frac{A}{\sin\theta\sqrt{mgd(\cos\theta - \cos\theta_0)\sin^2\theta + \frac{1}{2}mv_0^2(\sin^2\theta - \sin^2\theta_0)}}$$

and thus compute A.

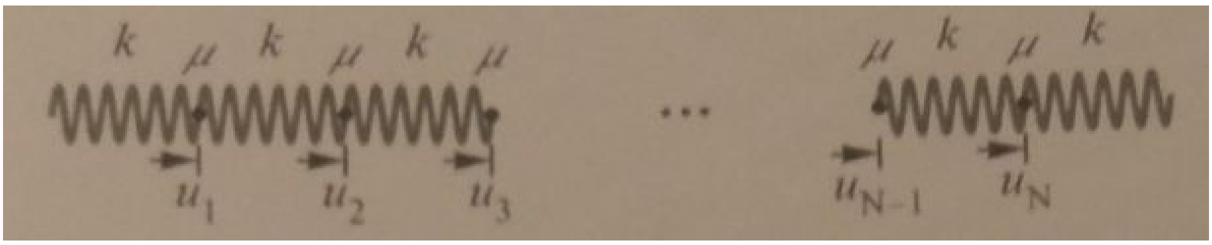
(E) If $\theta_0 << 1$, and $\theta_{min} = \frac{\theta_0}{2}$, find $\Delta\varphi$, the angle rotated about z-axis for the mass to go from θ_{min} to θ_0.

(F) If θ_0 is no longer a small quantity, but $\theta_0 - \theta_{min} = \alpha << 1$, find the value of $\Delta\varphi$.

A useful integral:

$$\int_{\frac{1}{2}}^{1} \frac{dy}{y\sqrt{(1-y^2)(y^2-\frac{1}{4})}} = \pi, \quad \int_0^{\sqrt{\alpha}} \frac{dz}{\sqrt{\alpha - z^2}} = \frac{\pi}{2}$$

5. The effect of mass on a spring *(Score: 20%)*



As the figure shows, view the spring as N point mass of mass μ, connected by springs of spring constant k, natural length l_0. Denote the position of the particles by $u_1, u_2, ..., u_n$, ignore any sideway motion.

(A) The spring above u_1 is hung to the ceiling, and let the surface of the ceiling be $z = 0$, with downward being the positive direction. Hang a point mass of mass M below u_N, and denote the position of the mass with u_{N+1}, list the equation of motion obeyed by the $N + 1$ particles.

(B) When the spring reaches equilibrium under the influence of gravity, $u_j = \eta_j = a_1 j + a_2 j(j + 1)$, find $a_1,\ a_2$.

(C) From (B), consider the continuity limit, $l_0 \to 0,\ \mu \to 0,\ j \to \infty,\ N \to \infty$, under this circumstance, the natural length of the spring, $L_0 = Nl_0$, mass $m = N\mu$, and spring constant $K = \frac{k}{N}$ will approach a constant. Let $s = jl_0$ be a continuous parameter, $0 < s < L_0$, and $\eta_j \to \eta(s) = b_1 s + b_2 s^2$, find $b_1,\ b_2$.

(D) From (C) , find the change in length of the spring ΔL, and the position of the center of mass ,X_0. Express your answer in terms of $m,\ M,\ g,\ K,\ L_0$.

(E) From (A), consider the continuity limit, $l_0 \to 0,\ \mu \to 0,\ j \to \infty,\ N \to \infty$, under this circumstance, let the density be $\lambda = \frac{m}{L_0}$, Young's modulus be $Y = KL_0$, find a differential equation of $u(s, t)$.

(F) Put the system on a smooth surface, and let $s = 0$ be the center of rotation with angular speed Ω, find a differential equation of $u(s, t)$ and its boundary condition.

(G) From (E), when the system is in a stable equilibrium, $u(s, t) = \eta(s) = A_0 sin(B_0 s)$, find $A_0,\ B_0$.

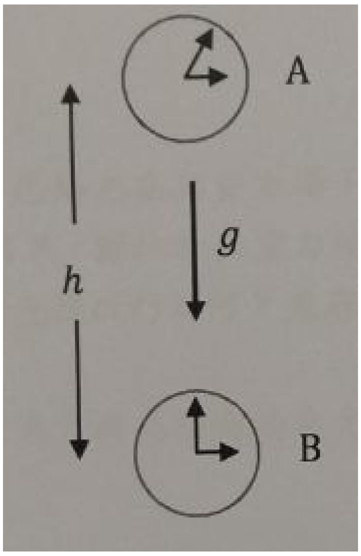
Mock APhO test - Time: 5 hours 150 marks total

1. Deflection of Light Under Gravitational Lenses (50%)

According the the principle of equivalence in General Relativity, the laws of physics in a uniform gravitational field g is exactly the same as the laws in a lab with the same acceleration.

In a non-uniform gravitational field, the statement can be generalized as follows: in a freely falling lab, the observed laws of physics are the same as the laws in special relativity, with no gravity. Free falling means the only external force during the motion is gravity.

(i) Consider the case of an uniform gravitational field g, the two clocks have a height difference of h, as shown in the figure.



It is known that the speed of light in vacuum is c, and A and B use the same clock (atomic clock). The clock is not affected by acceleration. Now, a light signal of frequency f_B is emitted from B, if we want to find f_A, the frequency A recieved, we can consider a free falling reference frame S. In S, both clocks are accelerating in the direction from B to A (opposite to the field). Suppose B was at rest relative to S when it emitted the signal (at time $t = 0$).

According to the principle of equivalence, in S, special relativity is applicable. It is known that in a gravitational field, when h is small enough, the shifting-ratio of the light wave can be denoted as:

$$\frac{f_B - f_A}{f_B} = \alpha \frac{gh}{c^2}$$

find α.

(ii) Suppose the same atoms, radiating the same frequency of light, are used to do the timing, then the frequency shifting in (i) can be viewed as the same clocks, but runs differently because of the different gravitational fields. If α is positive in (i), we can see it as the clock period at B, Δt_B, is longer than the period at A, Δt_A ($\Delta t_B > \Delta t_A$, time dilates at B), so the frequency of photons from B, received at A, its frequency f_A is smaller; otherwise, we can see it as the clock period at B, Δt_B, is shorter than the period at A, Δt_A ($\Delta t_B < \Delta t_A$, time contracts at B). Because special relativity is applicable to S in (i) therefore, time dilation must come with the same factor of length contraction. If the wavelength we found at A, B is Δl_A, Δl_B, respectively, then $(\Delta l_A / \Delta l_B) \times (\Delta t_A / \Delta t_B) = 1$.

let $\Delta t_B / \Delta t_A = \gamma$, and the speed of light at A, B is c_A, c_B respectively, what is c_A / c_B?

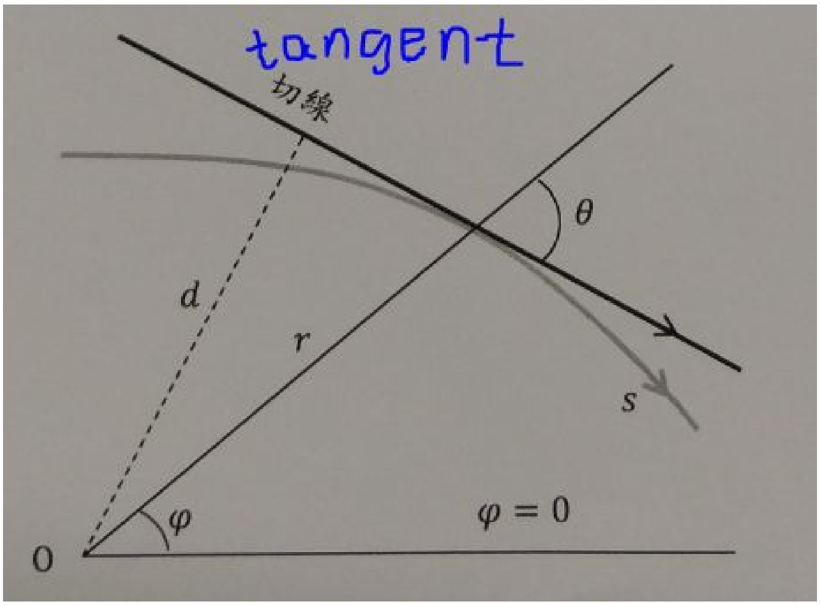
(iii) Suppose a planet can be viewed as a uniform sphere of mass M, radius R, and the canter of the sphere is the origin of polar coordinates. G denotes the gravitational constant, suppose $GM << Rc^2$. From the result of (i), if atoms on the surface of the planet radiates a photon of frequency f_R, then at the point of infinity (point A), the frequency received, f_∞ is equal to

$$f_\infty = (1 + \frac{k}{R}) f_R$$

find constant k.

(iv) From (ii) and (iii), it is known that the speed of light in zero gravity is equal to the speed of light in vacuum, c, then, from the results of (ii) and (iii), what is the speed of light where the distance to the center of the planet is r, $c(r)$? what is the refractive index $n(r)$ at r in that case?

(v) From (iv), let the angle between the direction of light and radial be θ, (r, ϕ) denotes polar coordinates in a plane, the path length along the trajectory of light is s, as the following figure shows.



According to Bouguer's formula, when the refractive index of a medium has a spherical symmetry, Snell's law should be restated as:

$$rn(r)sin\theta = n(r)d = const.$$

when the light shoots with impact parameter $b > R$ from infinity on the left, because of gravitational lensing effect (i.e. the deflection of light), when the light moves across the planet ($r \approx R$) and go toward infinity on the right, its direction will deflect for an angle $\Delta\phi$.

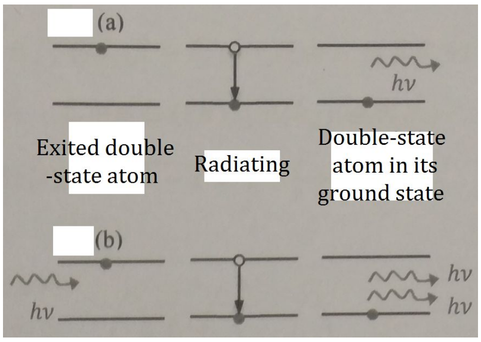
find the equation of the trajectory of light (i.e. find the equation $\frac{dr}{d\phi}$ satisfies), and use it to solve for $\Delta\phi$.

Note that the following mathematical formula may be of use:

$$\frac{d}{dx}sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2-x^2}}$$

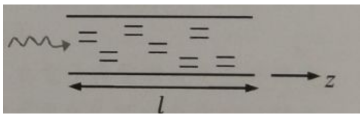
($sin^{-1}\frac{x}{a}$ is in the first or the fourth quadrant.)

2. Interaction between electron and light in free electron lasers (50%)

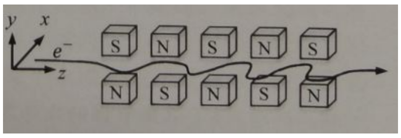


Light can interact with electrons when it's transmitting in a media, this will cause the intensity to change. In a microscopic view, there are two important processes, one is spontaneous emission, when electrons can emit photons spontaneously and return to ground state. Like how an excited double-state atom (with energy level $E_1 < E_2$, the energy of the photon is $h\nu = E_2 - E_1$) can emit a photon spontaneously as in Fig(a); the other is stimulated emission, when electrons in an excited state is influenced by photon of the same energy already existed in space and emit a photon, the probability of emission is proportional to not only the probability of electron transitioning, but also the intensity of light. Its reverse process is called stimulated absorption, and also satisfies the same relation. Answer the following questions.

(A) Planck's law can also be described as the energy density per unit frequency in an adiabatic cavity $u(\nu, T)$, when the temperature is T, $u(\nu, T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}$ where h is the planck's constant and k_B is the Boltzmann constant. Now consider there's also N double-state atoms as in Fig (a) in the cavity, suppose the probability of spontaneous emission is A, stimulated emission is B_{21}, stimulated absorption is B_{12}, when the system is in equilibrium at temperature T, deduce A/B_{21} and A/B_{12} from Planck's law.



(B) From (A), light of frequency ν and intensity I_0 shoots into a media containing double-state atom, suppose atoms at the lower energy state is N_1, atoms at the higher energy state is N_2, the distribution is uniform. The cross section of the media is a, the length is l, find the intensity out of the media, and find the condition at which the intensity can grow exponentially with l.

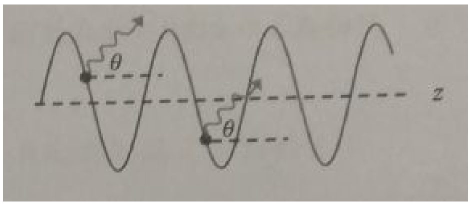


Apart from the aforementioned method, we can also use an external electric field and magnetic field to accelerate the electron, to let it interact with light waves and amplify it, the classical example is free electron laser, as the figure shows. Free electron laser uses alternating magnetic field(undulator) to form $\vec{B} = (0, B\sin(k_u z), 0)$ ($k_u = 2\pi/\lambda_u$, where λ_u is the period of the magnetic field in space), the field can accelerate incoming electrons to radiate light, then interact with electromagnetic wave to amplify it by stimulated emission. the initial position of the electron is $(0, 0, 0)$, initial velocity is v_0 in the z-direction, its mass is m, charge is $-e$, $\gamma_0 = (1-(\frac{v_0}{c})^2)^{-\frac{1}{2}}$, because the speed of the electron is very close to the speed of light c, $\gamma = (1-(\frac{v}{c})^2)^{-\frac{1}{2}} >> 1$, we need to account for relativistic effects. Answer the following:

(C) Find the ratio between transverse velocity and c, β_x and β_y, as a function of z, and thus find the average of β_z, $\bar{\beta}_z$. precise to $O(\gamma_0^{-2})$, express it with γ_0, K, where $K = eB/mck_u$.

Some helpful notes:

(1) The relation between the motion of the electron and time comes from the z coordinate of the electron.



(2) $\sqrt{1-x} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$

(D) From (C), because the effect of the alternating magnetic field, its x-coordinate varies periodically, at the same time, the electron radiates light because it's accelerating (this is spontaneous emission), this is similar to forced oscillation. The electromagnetic wave electron beam radiates because the undulator will be in resonance. It is known that for particles with speed close to c, the radiation is focused in a very small cone, therefore we can say the direction of radiation is constant. Consider the wave radiated by different electrons at different places, as the figure shows. The direction is θ. Suppose β_z can be approximated to be $\bar{\beta}_z$, and ignoring random initial phase of the wave, find the resonant wavelength λ_r, precise to $O(\gamma_0^{-2})$,when $\theta = 0$. Express it with γ_0, K.

(E) Now, the electrons are also influenced by the electric field it radiates. Suppose the electric field of the electromagnetic wave is $(E_0 cos(kz - \omega t + \phi),\ 0,\ 0)$, where $k = 2\pi/\lambda$. When the incident point is $z = 0,\ \phi = 0$, the interaction between the electron and the electromagnetic wave is as follows:

(i) The time varying rate of energy $\frac{d(mc^2\gamma)}{dt}$ is proportional to $\frac{d\gamma}{dt}$. It's known that when the z coordinate of the electron is z, $\frac{d\gamma}{dt} = A[cos(\Phi + \phi) + cos(\bar{\Phi} + \phi)]$, find $A,\ \Phi,\ \bar{\Phi}$.

(ii) the average of the time varying rate of electron (proportional to $\frac{d\gamma}{dt}$) with respect to coordinate z is an indicator of energy exchange between the electron and the electromagnetic wave. Suppose β_z can be approximated to be $\bar{\beta}_z$, and $\bar{\beta}_z$ in $\Phi, \bar{\Phi}$'s expression can be approximated to the case where there's no electromagnetic wave (i.e. $\gamma \approx \gamma_0$), precise to $O(\gamma_0^{-2})$, find the wavelength of the electromagnetic wave λ_m that can make the average of $\frac{d\gamma}{dt}$ with respect to coordinate z be maximum.

(F) For the beam of electron, phase of each electron's radiation is not necessarily equal, therefore if we want the maximum radiated electromagnetic field, we will have to synchronize each electron's radiation, the outputted light can then be at laser level. It's known that electrons within a wavelength is synchronized, and bunching occurs when energy is exchanged between the electron and electromagnetic wave, which make the electron density periodical in space with period λ_0. (i) Suppose the incident beam satisfies $z = 0,\ \phi = 0$, find $\frac{d\gamma}{dt}$ when the z-coordinate is z_0 relative to the canter. (ii) From that, deduce λ_0 of the radiated electromagnetic field when resonant condition is achieved, and explain how the bunching effect can synchronize radiated electromagnetic waves.

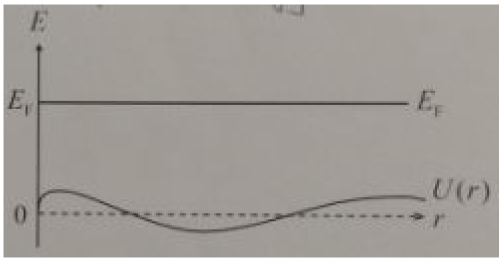
3. Shielding Potential of Charges in Metal and Conditions of Electron Bound State (50%)

Consider a proton of charge e formed in a piece of metal, and its contribution to electrons' potential $U(r)$. It's known that the potential of electron due to proton in vacuum is $U_c(r) = -\kappa_0 e^2/r$, and the number density of electron in metal is n_0, mass of an electron is m. First consider a metal without external charge, when the potential $U(r) = 0$. Because of Pauli's exclusion principle, each electron has to be in a different state. Electrons have two spin state, therefore each wave number vector $\vec{k} = (k_x, k_y, k_z)$ can contain two electrons:

(A) If the metal's measure in one direction (like x) is L, then the electron's wave function is the same at $x = 0$ and $x = L$, find the quantization condition of k_x

(B) If the maximum magnitude of electrons' wave number is $|\vec{k}|_{max} = k_F$, and n_0, k_F satisfies the relation $n_0 = ak_F^b$, find a, b. Hence express n_0 with E_F, the corresponding energy of k_F.

When there's external charges, $U(r)$ is no longer zero, but E_F stays constant, as the figure shows. When this happens, charge density will change with space. For the following parts, use k_F in place of n_0 in your answers.



(C) From the relation between n_0, E_F, introduce $U(r)$ and write the relation between $n(r)$ and E_F.

(D) Suppose $E_F >> U(r)$ and the change in space of $U(r)$ is sufficiently slow, so we can write $n(r) = n_0 + \delta n(r)$, then $\delta n(r) = -\alpha U(r)$, find α.

(E) Find the integration expression relating $U(r)$ and $\delta n(r)$.

(F) Replace $\delta n(r)$ with $U(r)$ and find the integration expression of $U(r)$.

(G) Multiply your result from (F) by r, and differentiate twice with respect to r to get $\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} = \beta^2 U$, find β.

(H) As a trial solution, substitute $U(r) = f(r)U_c(r)$ to the solution in (G) and find $f(r)$. $U_c(r) = -\kappa_0 e^2/r$ is the potential of an electron due to a proton in vacuum.

(I) Consider if $U(r)$ can bound an election: if r_0 is the radius of the electron's orbit in bound state, find the condition for the bounding energy to be negative.

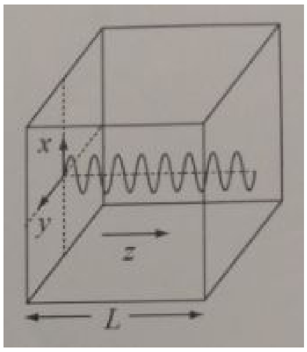
(J) Suppose we can change n_0 (or β) continuously, so that this quantum system only has one bound state. From how Bohr considers the hydrogen atom, find the expression of β under this condition.

Theoretical test 2 - Time: 4 hours, Total marks: 100

1. Quantum Optics *(Score: 20%)*

Light has wave-particle duality, so light waves can be described by one dimensional oscillator in classical mechanics. Quantum optics utilizes the quantization of Electromagnetic field to describe the quantum state of photons by quantum state of one dimensional oscillator.

Part A



Consider electromagnetic wave transmitting in a cubic cavity of side length L, and volume V, as the figure shows. The electric field is

$$E_x(z,t) = E_0 sin(kz) sin(\omega t + \phi)$$

where E_0 is the magnitude of the oscillation, $k = 2\pi/\lambda$ is the wave vector, ω is the angular frequency of the oscillation. Consider the case of $\phi = 0$ in parts (i) ~ (iii).

(i) the magnetic field in the cavity can be expressed as $B_y(z,t) = B_0 f(z,t)$. Find E_0/B_0 and the function $f(z,t)$

(ii) Find the total energy of the electromagnetic field in the cavity $E(t)$

(iii) Now introduce two functions $q(t)$ and $p(t)$ satisfies $q(t) \propto sin(\omega t)$, $p(t) \propto cos(\omega t)$. then $E(t)$ can be rewritten as $E = \frac{1}{2}(p^2 + \omega^2 q^2)$. Find $p(t),\ q(t)$, express them with $V,\ E_0,\ \omega$

P.S. Here, $p(t) = dq/dt$, $dp/dt\ =\ -\omega^2 q(t)$. The energy equation above can be viewed as the energy of an oscillator.

Part B

In quantum mechanics, energy of an one-dimensional simple harmonic oscillator can be written as

$$E_n = (n + \tfrac{1}{2})\hbar\,\omega$$

Here, n can be interpreted as the number of photons, $\hbar = h/2\pi$, h is the Planck's constant. q and p satisfies Heisenberg's uncertainty principle, meaning that

$$\Delta q \Delta p \geq \hbar/2$$

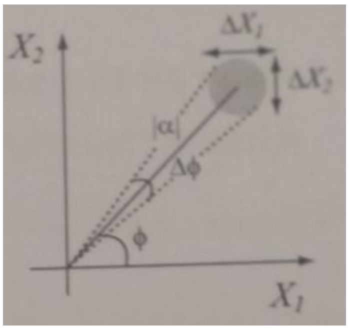
is true. Now, rewrite the electric field as

(iv) $E_x(z,t) = \sqrt{\frac{4\pi\omega}{\varepsilon_0 V}} E_0 sin(kz)\{cos\phi\, X_1(t) + sin\phi\, X_2(t)\ \}$ and $X_1(t) = \eta q(t)$, $X_2(t) = \xi p(t)$.
Find the parameters $\eta,\ \xi$.

In quantum optics we often use phasor diagram (i.e. $X_1 - X_2$ graph) to describe the quadrature of the electric field. The quantum state of photons α can be expressed as $\alpha = X_1 + iX_2$ in the phasor diagram as shown on the next page.

(v) Find $|\alpha|$, express it in terms of V, E_0, ω.

(vi) Consider the average energy of the electromagnetic wave, when $n >> 1$, we can get $n = |\alpha|^a$. Find the constant a. Strictly speaking, we have to consider the average number of photons in the cavity, meaning n is expressed by $\bar{n}$.



(vii) From the quantum energy equation above, the system has energy $\hbar\omega/2$ even when there's no photon inside the cavity, this energy is called zero-point energy. We can say that the existence of zero-point energy is caused by vacuum fluctuation. If that's the case, find the electric field fluctuation in vacuum $E_{vac.}$

(viii) In the phasor diagram, the quantum state of the photon satisfies $\Delta X_1 \Delta X_2 \geq \gamma$. Find γ.

(ix) The uncertainty of phasors can also be written as $\Delta n \Delta\phi \geq \zeta$. Find ζ.

This inequality means that photons have an uncertainty on particle number n and phase ϕ.

2. Hall Resistance *(Score: 20%)*

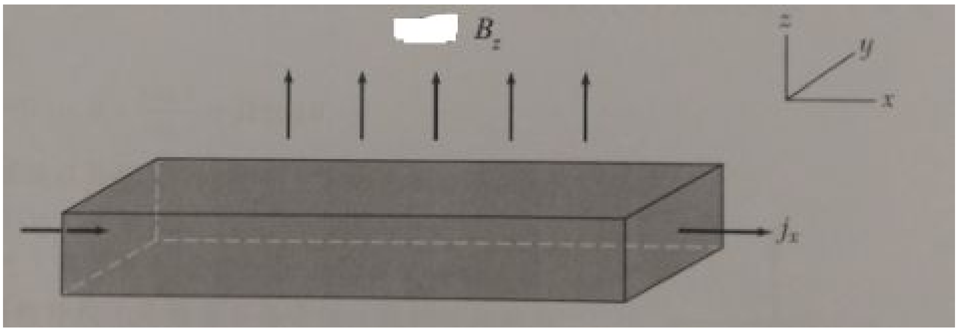
(A) Consider a metal plate in an uniform magnetic field $\vec{B}$ in the z-direction as the first figure shows. It is known that the equation of motion of charged particles in metal is $dp/dt = -\vec{p}/\tau + \vec{f}(t)$, where τ is a constant denoting the average collision time of the particle, $\vec{p}$ denotes the average momentum of the particle, and $\vec{f}(t)$ is the external force. Current density is defined as $\vec{j} = ne\vec{v}$, where $\vec{v}$ is the velocity of the particle, n is the number density of particles. If $\vec{j} = (j_x, j_y, j_z)$ is in a steady state, and $\vec{j} = \sigma\vec{E}$, so we can write $\vec{j}$ as:

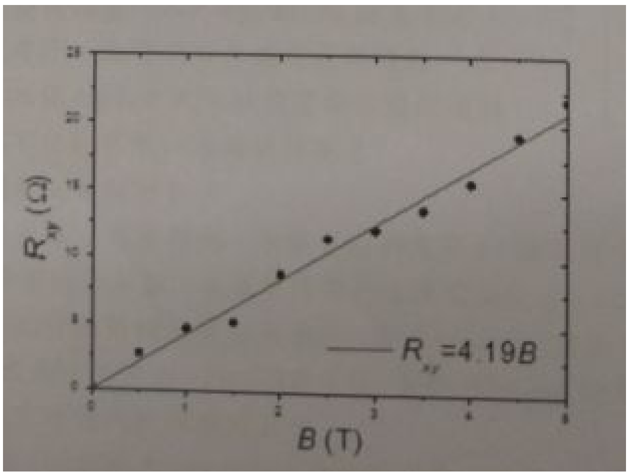
$$j_x = \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z$$
$$j_y = \sigma_{yx}E_x + \sigma_{yy}E_y + \sigma_{yz}E_z$$
$$j_z = \sigma_{zx}E_x + \sigma_{zy}E_y + \sigma_{zz}E_z$$

Find all the σ_{ij} terms where $i,j = x,y,z$. and express your answer with $\sigma_0 = ne^2\tau/m$, $\omega_c = eB/m$ and τ.

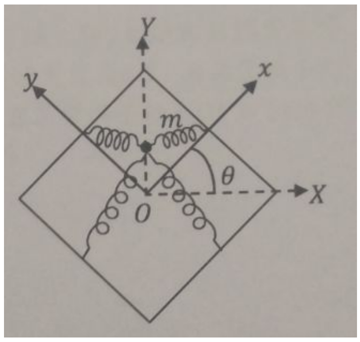
(B) Rewrite it as $\vec{E} = \rho\vec{j}$. Find all the ρ_{ij} terms where $i,j = x,y,z$.

(C) in Hall measurements, define the Hall resistance as $R_{xy} = V_y/I_x$, where V_y is the potential difference in the y-direction, I_x is the current passing. The second figure shows the data from a few Hall measurements. If the lattice constant of the metal is $0.3926\ nm$, find the number of charged particles per lattice.





3. The Resolution Power of Vibrational Gyroscope

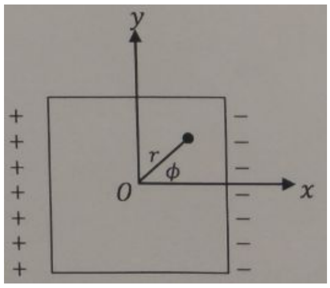


Gyroscope is an object used to detect and maintain the orientation of an system on Earth, it usually uses the law of conservation of angular momentum to maintain the orientation. Nowadays, vibrational gyroscopes are often used in electronics. The figure on the right is a simplified model of a vibrational gyroscope. A point mass m is put on a smooth, square platform of side length $2a$, the point mass is constrained by four springs of spring constant $m\omega_0^2/2$ to points $(a,0),\ (-a,0),\ (0,a),\ (0,-a)$, the vibration of the point mass can be used to detect the rotation of the platform, which can be further used to know the orientation (It takes 3 gyroscopes to fully know the orientation). Suppose the springs have negligible initial length, please answer the following:

(A) When the platform is not rotating, and in the xy coordinate system shown in the figure, the position of point mass is (x,y), write down the equation of motion of the point mass.

(B) When the platform is spinning with angular speed Ω, let the XY coordinate system be an inertial frame. at time $t=0$, XY and xy coincide , therefore $\theta=\omega t$. if the position of the point mass in xy coordinate system is (x,y), and (X,Y) in the XY frame, then $x=X\cos\theta+Y\sin\theta$, $y=-X\sin\theta+Y\cos\theta$. find the equation of motion in the rotating frame, and explain how the motion in the platform frame can be viewed as a point mass under the influence of a potential $V(x,y)$ (Let $V(0,0)=0$) and an uniform magnetic field B pointing straight out of paper(Let the equivalent charge of the point mass be 1). Find $V(x,y),\ B$.

(C) When the gyroscope is spinning with angular speed Ω, the motion of the point mass can be described with (x,y) or (r,ϕ), like the figure on the right shows(the springs aren't shown for simplicity). The angular momentum isn't conserved because of the equivalent magnetic field. We can generalize angular momentum is a certain way to make it become a conserved quantity, find the conserved quantity, express you answer with $r,\ d\phi/dt,\ m,\ \Omega$. for a certain angular speed the mass can perform stable circular motion, find the angular speed, express your answer with $r, m,\ \Omega,\ \omega_0$. When the point mass performs circular motion, under what circumstances can the motion be in one direction? Is it directed clockwise or counterclockwise?



(D) To make the gyroscope able to detect rotation, charge the point mass with charge q, and make the platform a capacitor, putting two charged metal plate on $x=\pm a$, to produce an uniform oscillating field $E_0\cos\omega t$, if we detect the motion of the mass on the y-axis, we can know if the platform is rotating or not. Suppose $\omega_0>\Omega$, and when the mass is moving relative to the platform, a damping force $\vec{f} = -2m\beta\vec{v}$ is exerted on the mass($\beta > 0$),the damping coefficient is small enough that the point mass can still perform oscillating motion.

(i) What is the trajectory of the stable motion?

(ii) If we wish to detect rotation, then the bigger the amplitude of the y-axis motion of the mass , the better it is. This can be done by changing the frequency of the electric field to resonance frequency. Ignore the contribution of β , and find the resonance frequency ω_R of the system, and explain how this can be used to detect rotation.

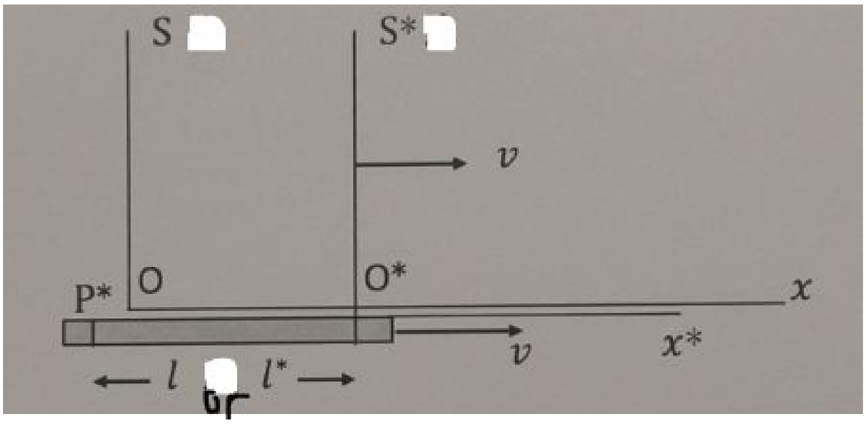
(iii) Whether we can detect the angular speed of the rotation depend on the quality factor of the resonance Q, if the width of the peak is $\Delta\omega$, then $Q = \omega_R/\Delta\omega$. Estimate the quality factor of each resonant frequency. For $Q >> 1$, find when it is possible to get the rotating angular speed from the resonant frequency.

4. Transformation between length and time

As the figure below shows, the origin of the inertial reference frame S is O. A rigid ruler has two marks on P^* and O^* , the origin of the inertial reference frame S^* is fixed at O^* , there are two observers, one is at O, one is at O^* . the x-axis of both coordinate systems are collinear, and the relative motion between the two frames is along the x-axis, the speed is a constant v , when O and O^* coincide, the time at the origin of the two frames are zero. The speed of light in vacuum is c, and let $\gamma = (1 - v^2/c^2)^{-1/2}$.

Part A

Fix the ruler in S^* . S^* move in the positive x direction with speed v, while S remains stationary. t is the time when P^* and O coincide, measured in frame S
Let l and l^* denote the length of P^*O^* measured in S and S^* respectively.



(i) find the ratio between l and l^*

(ii) Let t be defined as above, and t^* is the time when P^* and O coincide, measured in frame S^*, find the find the ratio between t and t^*

(iii) Let t be defined as above, what is $t^*{}_{O^*}$, which is what the clock on O^* reads at time t seen by the observer at O?

Part B

In this part, different to Part A, we let S^* be the stationary frame, and S is moving to the negative x direction with speed v. When O and O^* coincide, the time at the origin of the two frames are still zero

We know that in the situation described, the length of P^*O^* measured by S^* is $L^* = l^*$, answer the following:

(iv) In the situation described above, when the time on S^* is $\bar{t}^* = t^*{}_{O^*}$, what is the time $\bar{t}$ at O, the origin of S? $t^*{}_{O^*}$ is what we got in (iii), and is defined as above.

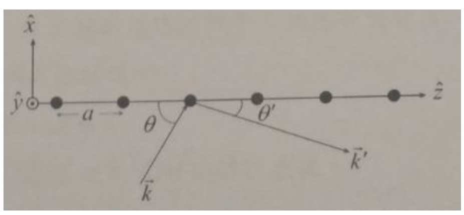
(v) what is the time T^* when O reaches P^*, measured by the observer at S^*? How about the time T when O reaches P^*, measured by the observer at S? When the time of S is T, what is the time $\bar{t}_{O^*}{}^*$ at the origin of S^*? If the measured length of P^*O^* is L, what is L^*/L?

(vi) when the time at S is T, the observer at O sends a signal to tell the observer at O^* that O has reached P^*. The signal travels with speed c' with respect to S^*. If we want the signal to reach before the time of S^* is T^*, what is the minimum value of c'?

5. Diffraction pattern of chiral material

Using diffraction pattern of tell left-handed and right handed molecules apart:

We first consider the diffraction pattern of one-dimensional lattice:

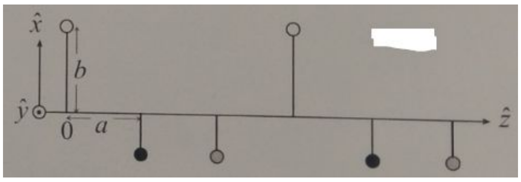


(A) The figure shows the incident wave (with wave number k), the incident angle is θ, the diffracted wave has wave vector $\vec{k'} = (k, \theta', \phi')$, find the condition on θ' for constructive interference to happen

(B) If the amplitude of the diffracted wave emitted from each lattice point is proportional to that of the incident wave, and the emitted wave is spherical, then the diffracted wave is $\psi_{inc}(\vec{R}_m)\{e^{ik|\vec{r}-\vec{R}_m|}/|\vec{r}-\vec{R}_m|\} \approx \psi_{inc}(\vec{R}_m)\{e^{ik|\vec{r}-\vec{R}_m|}/|\vec{r}|\}$ when $|\vec{r}| \gg \vec{R}_m$ is satisfied. Let $\psi_{inc}(\vec{r}) = e^{i\,\vec{k}\cdot\vec{r}}$, $\vec{R}_m = ma\,\hat{z}$, find the total diffracted wave ψ_D from N lattice point.

(C) Find the condition of constructive interference from ψ_D.

Consider the diffraction from one-dimensional left/right-handed symmetric lattice:
The figure on the right shows an one dimensional lattice with right handed symmetry, the small balls represent the atoms, it's the place that can produce diffracted waves. We ignore the diffracting effect of the connecting rods for the atoms, the thin rods have length b, and the neighboring rods are separated by distance a. also, as the z-coordinate increases, the thin rods rotate relative to each other with angle $\Delta\phi = \beta\phi_0$. $\beta = \pm 1$, $\phi_0 = 2\pi/3$. The figure shows the case when $\beta = -1$ (corresponding to clockwise rotation), different beta represents different chirality.



(D) the position of the atom is $\vec{R_m} = \vec{\rho_m} + ma\hat{z}$, find $\vec{\rho_m}$, β is not yet determined.

(E) Use $m = 3M + j$, where $j = 0, \pm 1$, $\vec{k}$ is the same as in **(A)**, $\vec{k'} = (k, \theta', \phi')$, find ψ_D. Find the condition of constructive interference from ψ_D.

(F) Consider the special case of $\theta = \pi/2$, $a = b = \lambda$, and the condition for constructive interference is fixed at first order, find the relation between the intensity of wave I ($I = |\psi|^2 = \psi\psi^*$) and ϕ'.

(G) Substitute $\phi' = \varphi + \pi/2$ into $I(\varphi)$, and prove that $I(\varphi)$ is an even function.

(H) Discuss the feature of I at $\varphi \approx 0$ (or $\phi' \approx \pi/2$) for $\beta = \pm 1$.
