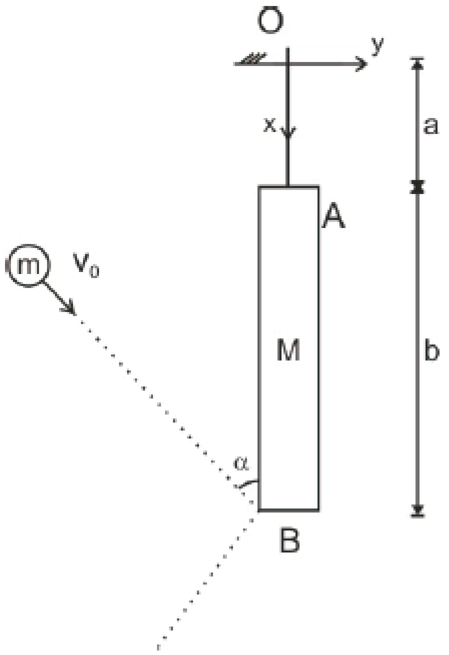
1. Rigid Body Collision

A small sphere of mass *m* and radius *r* collides with the far end *B* of a long homogeneous bar of mass $M=4m$ and length $b=9a$, as shown in the figure.



Consider that the collision is elastic, that the coefficient of friction between the sphere and the bar is $\mu=0.6$ and that the angle between the initial velocity v_0 of the sphere and the axis of the bar is α, as shown in the figure.

a) Determine, in terms of m, v_0 and α, the value of the impulses J and K from the sphere on the bar, in the y and x directions, respectively. What is the angle formed between the axis of the bar and the velocity of the sphere immediately after the collision? What is the condition of α so that the sphere moves upwards after the collision? (3 marks)

b) Determine the angular velocity ω_e of the sphere after the collision in terms of v_0, r and α. (1.5 marks)

c) Determine the angular velocity ω and the velocity of the center of mass v_{cm} of the bar right after the collision. Express the results in terms of v_0, b and α. (2 marks)

d) Determine the tension T in the rope (of length a) that holds the bar, immediately after the collision with the sphere. The local gravitational acceleration is g. Express the result in terms of m, g, v_0, a and α. (3.5 marks)

2. Electrostatic Analogy

The equations for many different physical situations have exactly the same looks. Obviously the symbols can be different (one letter substituted by another), but the mathematical form of the equations is the same. And the same equations have the same solutions! The equations of electrostatics, for example, appear in many other parts of physics, and, because of this, it is possible to solve problems of other areas with the same ease (or the same difficulty) of electrostatics.

The equations of electrostatics are:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad (1)$$

$$\vec{\nabla} \times \vec{E} = \vec{0} \qquad (2)$$

where ρ is the volumetric density of charge in a given point of space.

In this problem we will try to solve a situation of fluid dynamics with an electrostatic analogy. We need to highlight that this example isn't the best, because, for this to be possible, we have to consider a case almost hypothetical, making approximations and assumptions that rarely are valid when we study real fluids. The mathematician John Von Neumann once said that those who study the equations proposed next are studying "dry water".

Let us consider an incompressible, non-viscous and on a regime of non-turbulent flow. For an incompressible fluid we may write:

$$\vec{\nabla} \cdot \vec{v} = 0 \qquad (3)$$

And for a non-turbulent fluid, that is, in laminar flow (also called *irrotational*) we have:

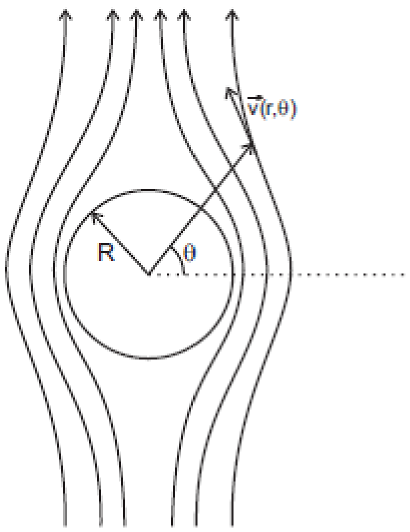
$$\vec{\nabla} \times \vec{v} = \vec{0} \qquad (4)$$

Note that these are the same equations that govern electrostatics for the free space (without charges, that is, where *ρ=0*). To verify this, see the table below:

Eletrostática (Espaço Livre)	Fluidodinâmica (conforme suposto)
$\vec{\nabla} \cdot \vec{E} = 0$	$\vec{\nabla} \cdot \vec{v} = 0$
$\vec{\nabla} \times \vec{E} = \vec{0}$	$\vec{\nabla} \times \vec{v} = \vec{0}$

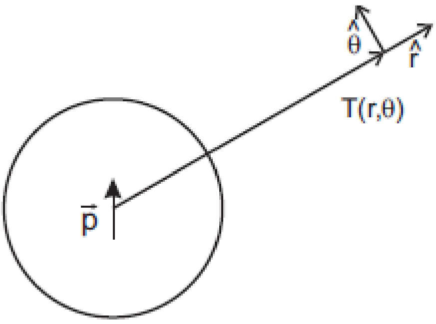
The problem: Consider a ball of radius R falling with a constant velocity (terminal) *v_0* in an incompressible and non-viscous fluid, in a non-turbulent way. If the sphere descends too slowly, the viscous forces, and are being neglected, will become important. If, however, it descends with great speed, swirls will appear (turbulence phenomenon) and there will be *circulation* of the liquid, where $\vec{\nabla} \times \vec{v} \neq 0$. So, we will have to focus on a regime where the ball has an intermediate velocity between these 2 extreme situations, in such a way that our suppositions are acceptable.

Put yourself in the reference frame of the ball, in such a way that you see water flowing (upwards) around it. We will use the electrostatic analogy to determine what is the velocity of the liquid $\vec{v}(r, \theta)$ in every point of space!



a) We can consider as an electrostatic analogy a system where a sphere of radius R is immersed in a region that has a uniform electric field. In this case, what would theoretically be the dielectric constant *κ*, or the relative permittivity *ε_r*, of the sphere? (2 marks)

b) Assume that the electric field generated by this sphere, for $r \geq R$, is equivalent to the electric field produced by a punctual electric dipole located in the center of the sphere. Determine the electric field $\vec{E}_e$ generated by the sphere, in polar coordinates, in a point $T(r,\theta)$ as a function of the electric dipole moment $\vec{p}$ (that points in the upwards vertical direction). Consider that the medium where the sphere is located has a permittivity *ε=ε_0*. (2 marks)



Obs: $\hat{r}$ and $\hat{\theta}$ indicate the versors in the radial and angular directions, respectively.

c) Calculate the total field $\vec{E} = \vec{E}_e + \vec{E}_0$, where $\vec{E}_0$ is the uniform electric field in which the sphere is immersed, and points vertically upwards (the same direction of the vector $\vec{p}$ in the figure. (1 mark)

d) Determine the value of $|\vec{p}|$. (2.5 marks)

e) Calculate $\vec{v}(r,\theta)$ in polar coordinates (that is, utilizing the versors $\hat{r}$ and $\hat{\theta}$ and the quantities r and θ) for $r \geq R$, in terms of the modulus of the descent velocity *v_0* and the radius *R*. (2.5 marks)

Obs: Vector field written in the cartesian system:

$$\vec{E} = E_x(x,y,z)\hat{x} + E_y(x,y,z)\hat{y} + E_z(x,y,z)\hat{z} \quad (5)$$

- Divergent operator of a vector field:

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (6)$$

- Rotational operator of a vector field:

$$\vec{\nabla} \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z} \qquad (7)$$

3. Relativistic Rocket

a) (Classical Case) Consider a rocket free of external forces. It has an initial velocity *v_i* and initial mass *m_i*. Suppose that it ejects fuel at a constant velocity *u*, relative to the rocket, in the opposite direction of the initial velocity. Calculate the velocity of the rocket as a function of its mass. (2 marks)

b) Now, suppose that the rocket can reach relativistic speeds. Suppose, as in the previous question, that there are no external forces on the rocket. Suppose that the initial proper mass of the rocket is m_{0i}, that the rocket is ejected at a constant velocity *u* relative to the rocket and that it starts from rest.

Calculate the velocity of the rocket, as seen by a stationary observer, external to the rocket, as a function of the proper mass of the rocket. (4 marks)

c) Now consider that the rocket is moved by photons, that is, instead of ejecting fuel, it ejects photons of frequency *f_0* (relative to an observer in the rocket). Let m_{0i} be the initial mass of the rocket. Calculate *dv/dn*, where *v* is the velocity of the rocket and *n* is the number of photons emitted by the rocket, as a function of *v*, *n*, m_{0i} and the necessary physical constants. (4 marks)

If necessary, use:

$$\int_0^x \frac{dx'}{1 - \frac{x'^2}{a^2}} = \frac{a}{2}\left[ln\left(\frac{1 + \frac{x}{a}}{1 - \frac{x}{a}}\right)\right] \qquad (8)$$