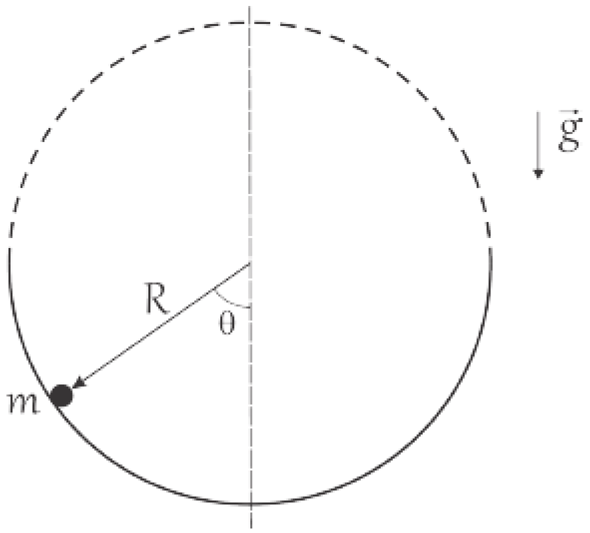
1. Movement of a particle on a spherical surface

Consider a particle of mass m placed on a spherical surface of radius R. Let θ be the angle, relative to the vertical, of the position of the particle in a given instant of time. See the figure below.



The movement of the particle is considered only in the plane of the paper.

Suppose that the particle is placed, initially, in an angle θ_0 and with angular velocity $\omega_0=(d\theta/dt)_0$. Also assume that the movement is always restricted to the region $-\pi/2 \leq \theta \leq \pi/2$, so that we won't have to worry about the particle losing contact with the surface.

a) Calculate the angular velocity ω of the particle when it is located in a given angle θ. Express the result in terms of ω_0, θ_0, the radius R and the local gravity g.

b) To what intervals the angle and angular velocity are restricted to in this case?

Now we assume there is a coefficient of dynamical friction $\mu(\theta)$, that can depend on the point, but is a continuous function of the angle.

Suppose that the particle doesn't roll, it only slips. In the next questions, we will answer the questions a) and b), but including the effect of friction:

c) Write the differential equation that relates the angle *θ* of the position of the particle with the time *t*. Make it depend only on known parameters (given in the problem text). Remember that, due to the fact that the direction of the friction force depends on the direction of the velocity, we will need *two* equations, for the cases where the particle is moving clockwise or counterclockwise.

After it is released from the initial position *θ_0>0*, starting from rest, it will start to descend along the surface until it stops. Then, there are many possibilities, depending on the level of friction: it is possible that is stops in a positive angle, and, if the static friction can handle it, it will remain stationary; it is possible that it stops in a negative angle (that is, it goes beyond the lowest point of the cavity) but remains stationary, if the static friction is enough; or it can also reach a negative angle and descend once again, in the counterclockwise direction, and then any of the three situations will repeat. However, we will analyse only the first part of the motion, when the particle moves in the clockwise direction (before the first stop).

It may be possible to realize that it is pretty complicated to directly solve the equation of question c). However, we will see that it will be much easier to solve the angular velocity as a function of the angle (*ω=ω(θ))* and then, at first, it will be possible to determine *θ(t)*, or better, *t(θ)*, with the integral

$$t = \int_{\theta_0}^{\theta} \frac{d\theta}{\omega(\theta)}$$

d) Show that it is possible to relate the angular velocity with the angle with the following differential equation:

$$\frac{d\omega^2}{d\theta} - 2\mu\omega^2 = -\frac{2g}{R}(\sin\theta - \mu\cos\theta)$$

Such an equation is much simpler to solve (it is known as a *first order linear differential equation*). The idea to solve something like this is to try to use the chain rule to group the incognito function (in this case *ω^2*) in one single derivative, so we can trivially integrate the equation. Take a look at a similar equation where we see a direct application of the ideia above:

$$f(x)\frac{dy}{dx} + f'(x)y = J(x) \Longrightarrow \frac{d[y(x)f(x)]}{dx} = J(x)$$

where $y=y(x)$ is the incognito function, x is the independent variable and f, J are known functions.

e) Solve the differential equation of the previous question. Don't forget that μ isn't constant. *Hint*: Try to multiply the equation by a function $\lambda(\theta)$ so you can group terms according to what was shown.

f) What changes must be made to the solution obtained above to analyse the case where the particle moves in the counterclockwise direction?

Now let us make a simplification, sufficiently reasonable, that the coefficient of friction μ is constant on all of the surface.

g) Add this simplification to the previous obtained solution, obtaining the square of the angular velocity $\omega^2(\theta)$ as a function of the angle θ. This expression becomes that of question a) when we have $\mu=0$?

Hint: You may want to use the result of the following integral:

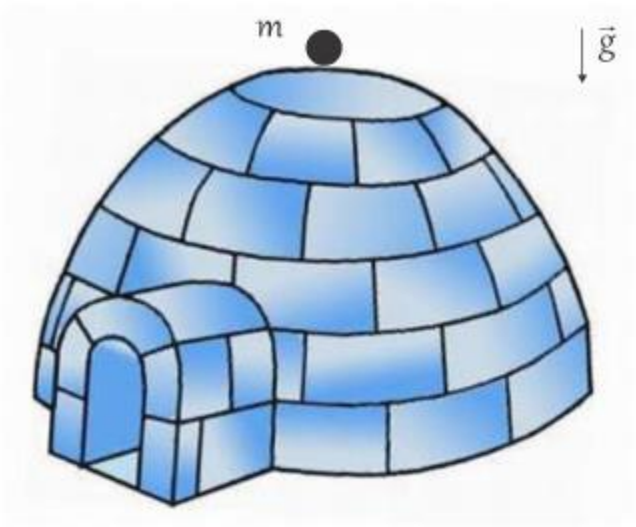
$$\int e^{-2\mu\theta}(\sin\theta - \mu\cos\theta)d\theta = -\frac{e^{-2\mu\theta}}{1+4\mu^2}\left[\left(1-2\mu^2\right)\cos\theta + 3\mu\sin\theta\right]$$

Aiming to draw some quick conclusions, without having to resort to numerical methods, and that help us to understand the effect of friction on the motion of the particle, even though we already have a very clear idea of this, suppose that the friction is very low, that is, the surface is almost smooth.

h) Show that, for every point of the surface, the angular velocity is lower compared to the case where friction is absent.

i) After releasing the particle from θ_0, and from rest, it will slide until it stops for the first time, on an angle θ_1. Since a bit of its energy was removed by the friction it won't be able to go as far as it would if there wasn't friction. Calculate the angle $\delta\theta$ that the particle is incapable of reaching, compared to the case without friction.

j) Now, consider that the surface where the particle descends is convex, instead of concave, such as an igloo. What changes must be made in the formulas to adapt them to this case?



k) Imagine someone sliding on the surface of this igloo, of coefficient of friction $\mu<<1$, that starts from the top and with almost zero initial velocity. Calculate the angle θ_*, in first approximation, where there is loss of contact.

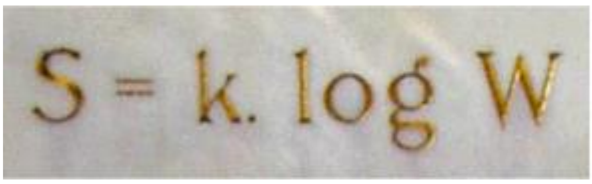
2. Schottky Effect

Let's consider a simplified model of an ideal gas composed by N particles that can be found in two states, with energies 0 or $\varepsilon>0$. To specify the microscopic state of this system, the knowledge of the number of particles in each energetic state is necessary. Consider the case where N_1 particles are in the 0 energy state and $N_2=N-N_1$ particles are in the energy state $\varepsilon>0$.

a) Considering that all the particles are identical and that the only form to distinguish them is by their energy, find in how many ways W it is possible to obtain a state like that described in the text, as a function of N, N_1 and N_2.

b) Express the result obtained in the previous question as a function of the total energy $E=\varepsilon(N-N_1)$ of the system, the energy ε and the total number of particles N of the sample.

The entropy of a system is given by Boltzmann's formula, expressed in his tombstone as shown in following figure:



(In this figure, “log” means the natural logarithm, of base *e*.)

c) Using the given formula, write the entropy S of the system composed by the two-levels gas.

In general, when we treat thermodynamic systems, we are interested in the properties for great populations, ie. when N, N_1 and N_2 are great numbers, of the order of 1 mol. In this case, we can use approximations that enable an analytic treatment of the problem without losing the basic characteristics of it. These approximations are part of what we usually call the *thermodynamic limit*.

One of the main approximations used is the famous Stirling expansion, that is given by

$$\log(x!) = x\log x - x + \mathcal{O}(\log x)$$

where terms of the order of $log(x)$ can be neglected in the thermodynamic limit.

d) Use the Stirling expansion to express the entropy density of the system $s=S/N$, as a function of the Boltzmann constant, the energy ε and the energy density $u=E/N$ of the system.

e) Obtain the temperature T of the system as a function of k, u and ε. If necessary, use:

$$\frac{d}{dx}\log f(x) = \frac{f'(x)}{f(x)}$$

f) With the results obtained in questions d) and e), sketch the graph of s as a function of u.

g) What is expected about the temperature when $u>\varepsilon/2$? Discuss the result.

h) Determine the energy density u as a function of ε and the Boltzmann factor $\beta=1/(kT)$.

i) The same result could have been obtained using the Boltzmann factors $P(0)$ and $P(\varepsilon)$, and a weighted average, that is, $u = 0 \cdot P(0) + \varepsilon \cdot P(\varepsilon)$. Determine which are these Boltzmann factors.

j) Sketch the graph of u as a function of the temperature T.

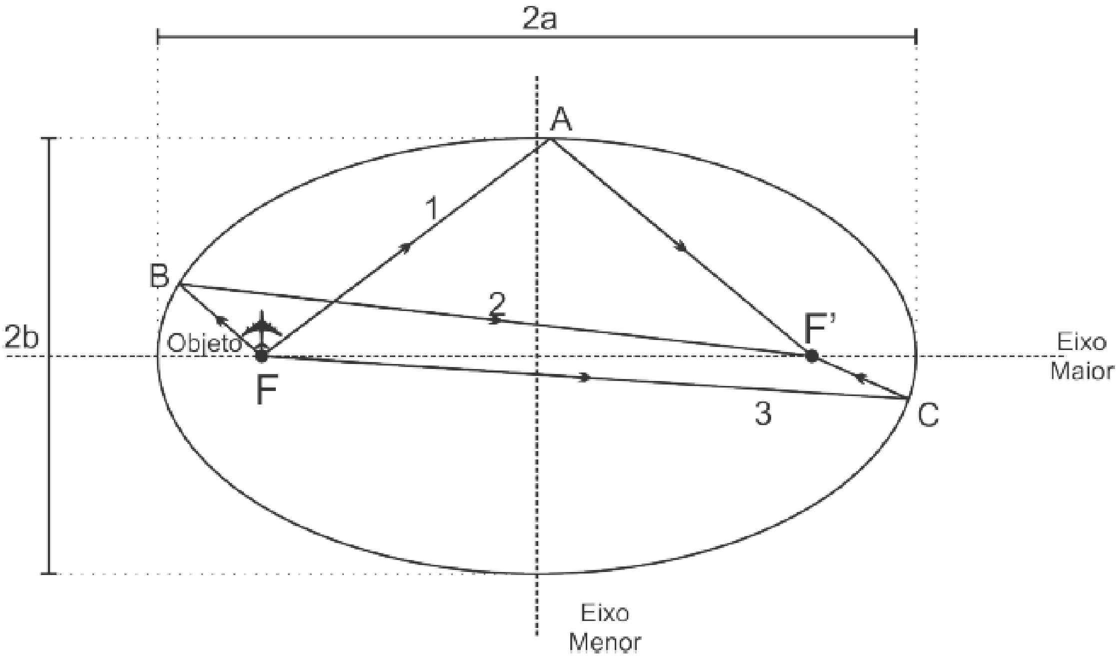
k) Find the specific heat c of the system as a function of β, ε and k.

l) Make a graph of the specific heat c as a function of the temperatura for this kind of system. Highlight in your sketch the limits $\beta\varepsilon\to 0$ and $\beta\varepsilon\to\infty$.

The signature of this kind of system is the graph of the specific heat obtained in the previous question, and is known as the *Schottky effect*.

3. Ellipsoidal Mirror

In this problem we will investigate the characteristics of the image formed by a mirror in the shape of an ellipse. This kind of mirror has the memorable property that every ray that starts from one of the foci *F* or *F'*, as shown in the figure, hits the other focus. In this case, we say that the foci *F* and *F'* are an object-image pair, or conjugated points.



The ellipse has major axis *2a* and minor axis *2b*.

An ellipse is characterized by its foci *F* and *F'*. The main characteristic of an ellipse is that for many point *P* on it, such as the points *A*, *B* and *C* on the figure, the relationship $\overline{PF} + \overline{PF'} = 2a$ is valid. We define the eccentricity ε of an ellipse as

$$\epsilon \equiv \frac{\sqrt{a^2 - b^2}}{a}$$

This quantity indicates how oval the ellipse is. Suppose that an object is placed on the focus *F* of the mirror, as shown in the figure. Answer the following.

a) What is the transverse magnification of the mirror if we consider only the light rays that hit the mirror near point *A*? Make an illustration of the formed image. Express your result in terms of the eccentricity ε of the ellipse. (Obs: The transverse magnification is defined as the ratio between the length of the image and the length of the object when the object is placed in the direction perpendicular to the optical axis of the mirror).

b) Repeat what was done in question a) considering the rays that hit the mirror near point B.

c) Repeat what was done in question a) considering the rays that hit the mirror near point C.

d) Based on the previous questions, determine what would be the transverse magnification of an ellipsoidal mirror.

e) The circle is a particular case of ellipse, where $\varepsilon=0$. Determine the transverse magnification of a spherical mirror, for an object placed on point *F*. Where is the image in this case?