1. The Motion of a Top

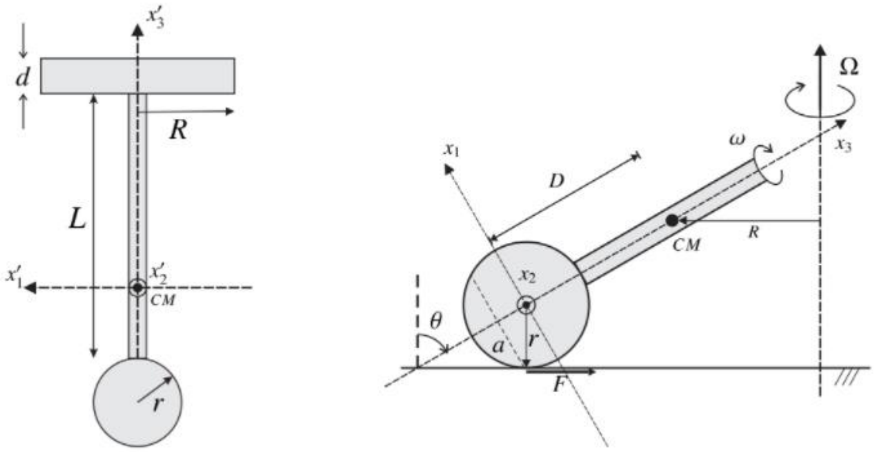
The most fascinating aspect of tops is the fact that they can temporarily *challenge* gravity moving back and forth, and rising and descending until they eventually fall. Also, if it can spin fast enough, it can stay in a stationary position where its axis of rotation remains vertical.

In this problem, we will investigate the model of a top that has a spherical tip and that performs a movement of precession about a fixed axis. In the end, we investigate a rough model in the case where friction is considered.

1.1 Rolling without Friction

Consider the top shown in the figure below. It is composed by a sphere of radius *r* in its base, by a disc of radius *R* and thickness *d* and a very thin bar of length *L* and transverse area *A*. Each element is glued one above the other, as shown below.

Consider that all the top is made from a material of density ρ.



In the first figure (1a), we have a diagram of the top. In the second one (1b), a movement of precession of the top with the main axis *x_2* pointing outwards the plane of the paper.

a) Determine the distance D of the center of mass (CM) of the top to the center of the sphere.

b) Determine the moments of inertia I_1, I_2 and I_3 of the top relative to its main axis x'_1, x'_2 and x'_3 shown in figure 1a.

Consider figure 1b that shows only the sphere and all the rest of the top is indicated by a bar. The top performs a movement of precession where the CM spins around a circle of radius R. The point of contact between the sphere and the ground spins around a circle of radius a. The angular velocities of precession are indicated in figure 1b.

c) Considering that the radius of the sphere is small, so that the main axis can be considered those shown in figure 1b, determine the component ω_3 of the top's angular velocity in axis x_3 and the angular momentum L_3 in this axis.

d) Determine the friction force F needed to maintain the movement.

e) Write the equation of motion of the top in axis x_2.

f) What is the condition between ω and Ω so there is pure rolling?

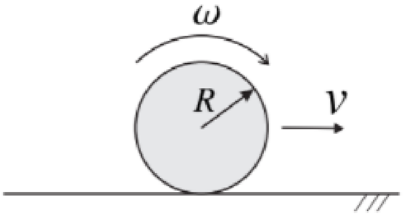
g) What is the condition on ω so that the movement is possible?

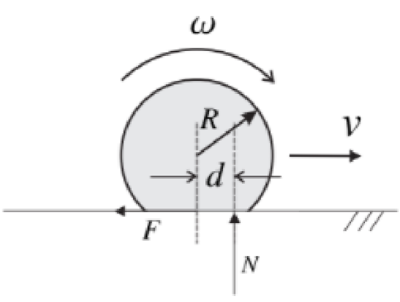
h) Determine the value of the angular velocity of precession Ω as a function of the angle θ, the angular velocity ω and the other important quantities in this problem.

i) Investigate the case where ω>>Ω.

1.2 Rolling with Friction

In this part of the problem we will make a quick visit to the problem of the rolling of a body when we consider the force of friction.





In the figure to the left (2a), we have the classical model of a cylinder rolling without slipping. In the figure to the right (2b), we have a more accurate model.

j) According to figure 2a, determine what is the condition of pure rolling for a cylinder of radius R when its CM moves forward with velocity *v* and the angular velocity about the CM is *ω.*

When a friction force F acts on the cylinder, for figure 2a, it tends to modify both the velocity of the CM and the angular velocity of the cylinder.

k) Indicate in fig 2a where the friction force acts and show that if the torque of this force is considered this becomes incompatible with the pure rolling condition obtained in j).

To fix the problem found in the previous question, we can adopt the scheme shown in figure 2b, where the normal force on the cylinder has a horizontal displacement relative to the CM.

l) At what distance d should be the normal force N so that the pure rolling condition is satisfied?

m) What must be the minimal value of the coefficient of static friction μ for the motion to occur?

2. Diamagnetic Levitation

Apart from our intuition, substances that apparently are non-magnetic can levitate in the presence of magnetic fields. The majority of substances are weakly diamagnetic and small forces associated with them makes levitation possible. Living creatures consist basically of diamagnetic substances (like water and proteins) and can be levitated or feel weak “gravities”.

With this in mind, the physicists *Andre Geim* and *Michael Berry* levitated a frog, as shown in the figure above. They won the Ig Nobel, a prize that is given to people that obtain scientific achievements that make people **laugh** and then **think** about science. A curious fact is that the physicist *Andre Geim* won the Nobel prize in Physics in 2010 for his job with *grafen*, a perfectly bidimensional stable sheet formed by carbon atoms.

2.1 Energy

Consider a solenoid as shown in Figure 4. The magnetic field in position $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\vec{B}(\vec{r})$ with magnitude given by $B(\vec{r}) \doteq |\vec{B}(\vec{r})|$, and the gravitational acceleration is *g.*

The object to be levitated has mass M and volume V (density ρ=M/V) and susceptibility χ.

The magnetic moment $\vec{m}(\vec{r})$ of a body submitted to a magnetic field $\vec{B}(\vec{r})$ is given by the expression

$$\vec{m}(\vec{r}) = \frac{\chi V \vec{B}(\vec{r})}{\mu_0}$$

where μ_0 is the magnetic permeability of vacuum and χ is its magnetic susceptibility.

a) What is the sign of the susceptibility of a diamagnetic body?

Suppose that the magnetic field on the body is slowly increased from zero to $\vec{B}(\vec{r})$.

b) Find the work performed on the body for it to acquire a magnetic moment $d\vec{m}$, as a function of the field $\vec{B}(\vec{r})$ to which it is submitted.

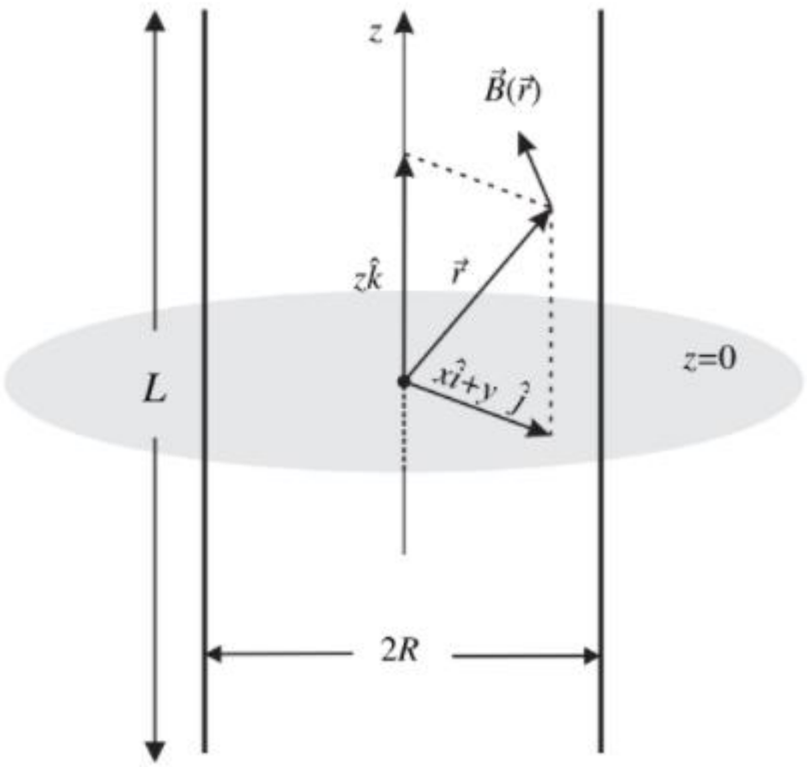


Figure 4: Solenoid. Inside of it, there is a diamagnetic body that will be levitated.

c) Considering the gravitational potential energy *m*g*z* write down the expression for the total energy of the body.

d) Find the net force on the body as a function of the data of the problem and of $\vec{\nabla}\vec{B}(\vec{r})$.

e) Supposing that the magnetic field depends only on *z*, that is, $\vec{B}(\vec{r}) = B_z(z)\hat{k}$, find the condition on $B(z)B'(z)$ so that the body remains in equilibrium.

2.2 Stability Conditions

Earnshaw's theorem says that it is not possible for a particle to remain in stable equilibrium only due to forces that scale with the inverse of the square of the distance, that is, 1/r^2. The same result is valid for permanent magnets (ferromagnets) and paramagnetic bodies, but this is not true for diamagnetic bodies.

Suppose that in a region of magnetic field $\vec{B}(\vec{r})$ there is a magnetic body with magnetic moment given by $\vec{\mu} = \eta\vec{B}(\vec{r})$.

f) Starting from the expression for the interaction energy between the magnet and the magnetic field, find what is the condition on $\nabla^2\vec{B}^2(\vec{r})$ as a function of η for the body to remain in stable equilibrium in the position $\vec{r}$.

g) What is the condition on $\partial_j^2\vec{B}^2(\vec{r})\ (j = x, y, z)$ for stability to happen in the vertical direction? And the horizontal? Write your answer as a function of η.

If we consider the cylindrical symmetry of the solenoid, it is possible to obtain relationships between the derivatives of the magnetic field both in the radial direction $\hat{r}$ and the vertical direction $\hat{k}$.

Suppose that the field is given by

$$B_z = B_0 + B_1 z + \tfrac{1}{2} B_2 z^2 - \tfrac{1}{4} B_2 r^2 + \cdots$$

$$B_r = -\tfrac{1}{2} B_1 r - \tfrac{1}{2} B_2 r z + \cdots$$

where $B_1 = \dfrac{\partial B_z}{\partial z}, \quad B_2 = \dfrac{\partial^2 B_z}{\partial z^2}$

with the derivatives taken in the point where the body is levitating.

h) Verify if the equation $\vec{\nabla} \cdot \vec{B} = 0$ is satisfied with the given terms.

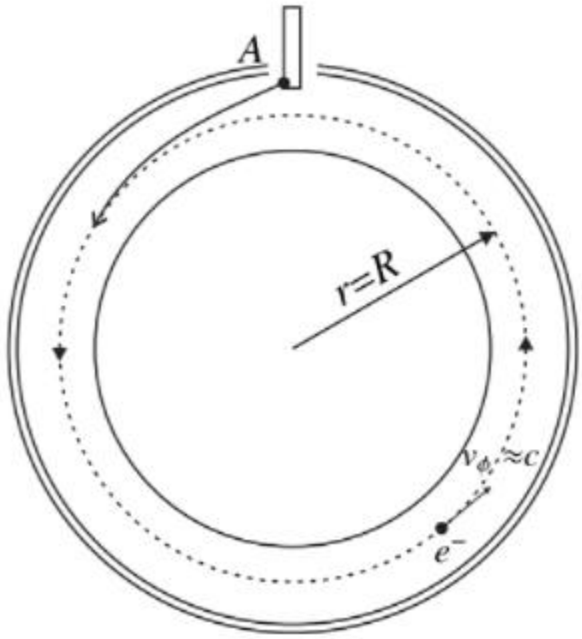
Note: Remember that the cylindrical symmetry has to be considered.

i) Re-write the equilibrium conditions both in the vertical and the horizontal directions as functions of B_0, B_1 and B_2 only. In this question, consider that the body has mass *m* and magnetic moment $\vec{m} = \chi V \vec{B}/\mu_0.$. The local gravity is *g.*

3. The Betatron

A betatron is an equipment that can be used to accelerate charged particles to high speeds with the application of a variable magnetic field. In this problem we will investigate the acceleration of *ultrarelativistic* electrons (electrons with speeds that approach the speed of light in vacuum *c*) under the influence of a magnetic field.

The betatron allows the electrons to be accelerated in a circular orbit of constant radius R varying only the flux of the magnetic field on the circle described by the electron. A scheme of this equipment is shown in the below figure.



In point A, electrons are ejected from a filament and then they travel through a circular orbit of radius R. As the magnetic flux in the circle varies, the speed of the electron rises.

3.1 Maximum Velocity of the Electron

Suppose, unless stated otherwise, that the treated electrons are ultrarelativistic, that is, their velocity is close to *c*. Consider the figure 6 that shows the movement of the electron in the betatron.

a) Determine the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ of the electron as a function of its orbital radius R and the magnetic field B_z perpendicular to the plane of the orbit and in the position r=R, where the radius r is measured relative to the center of the electron orbit. If necessary, use basic natural constants such as the charge *-e* and the mass *m_e* of the electron and the speed of light in vacuum *c.*

b) Determine the electric field E_ϕ tangent to the orbit of the electron as a function of basic constants and of the temporal variation

$$\dot{\bar{B}}_z = \frac{d}{dt}\bar{B}_z = \frac{1}{\pi R^2}\frac{d}{dt}\int_0^R B_z(r')2\pi r' dr'$$

of the average magnetic field measured in the circle of the electron's orbit, and also of the radius R of its orbit.

c) Determine what must be the relationship between the magnetic field B_z in the electron's orbit and the average value $\bar{B}_z$ in the circle described by the electron.

d) For example, show that if the field is given by

$$B_z(r) = B_0 \left(1 - \frac{2}{3}\frac{r^2}{R^2}\right),$$

it will satisfy the condition obtained in the previous question.

Suppose that the electron radiates energy according to Larmor's formula:

$$\frac{dE}{dt} = -\frac{e^2 a^2}{6\pi\varepsilon_0 c^3}$$

where *a* is the acceleration felt by the electron in its rest frame.
The average power doesn't change when we observe it from the lab's frame, although the acceleration of the electron changes.

e) Determine the acceleration a_L suffered by the electron relative to the lab
s frame if the acceleration in the rest frame is *a*. Write a_L as a function of *a* and *γ*.

f) Determine what must be the maximum value of γ obtained by the electron during the acceleration. Write your answer in terms of the temporal variation $\dot{\bar{B}}_z$ of the average field in the circle of the orbit, of the magnetic field B_z in the orbit, of the radius R of the orbit and of fundamental constants.

3.2 Stability Condition

Let us now not consider that the electron radiates energy during acceleration. Still consider that we are dealing with ultrarelativistic electrons.

We will investigate how the electron will move if it is slightly disturbed from its equilibrium orbit when the small displacements are in the *z* direction or the radial direction. Consider that there is displacement only in one direction at a time (for example, when it is disturbed in the radial direction it remains in z=0).

Consider that the magnetic field to which the electron is submitted can be written as

$$B_0(t) = B_z(R, 0, t)$$

Also define the gradient index of the magnetic field, *n*, as

$$n = -\frac{R}{B_0(t)} \frac{\partial B_z(R, 0, t)}{\partial r}$$

that indicates the variation of the *z* component of the magnetic field that acts on the electron.

g) Write the radial equation of motion for the electron as a function of $B_0(t)$, $\partial B_0(t)/\partial r$ and of fundamental constants.

Let $x \equiv r - R \ll R$ be the displacement of the electron in the radial direction, relative to its equilibrium position.

h) Find the equation of motion of *x* as a function of the angular frequency $\omega_0^2 = c/R$ of the electron, of B_0, of the radius R of its original orbit and of $\partial B_0(t)/\partial r$.

i) What is the condition on the index *n* for the orbit to be stable?

Suppose now that the electron is slightly displaced in the *z* direction, keeping the same radius *R* of its orbit.

j) Write the equation of motion for *z* as a function of $B_0(t)$ and of $B_r(R, z, t)$, and also of fundamental constants.

k) Knowing that in cylindrical coordinates

$$\nabla \times \vec{B} = \hat{r}\left(\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\phi) - \frac{\partial B_r}{\partial \phi}\right]$$

is null in the plane of the electron’s orbit, show that

$$B_r(R, z, t) = \int_0^z \frac{\partial B_z(R, z', t)}{\partial r} dz' \approx z \frac{\partial B_z(R, 0, t)}{\partial r}$$

l) Re-write the result of question j) and determine the condition on the index *n* for the orbit to be stable.

m) Determine the period of the motions both in the radial direction and in the vertical direction as functions of *n* and ω_0. Show that this period is always larger than the orbital period of the electron in the betatron. Also determine what is the condition on *n* so that both movements (*z* direction and radial) have the same period.