1. Elastic Collision Between two Particles

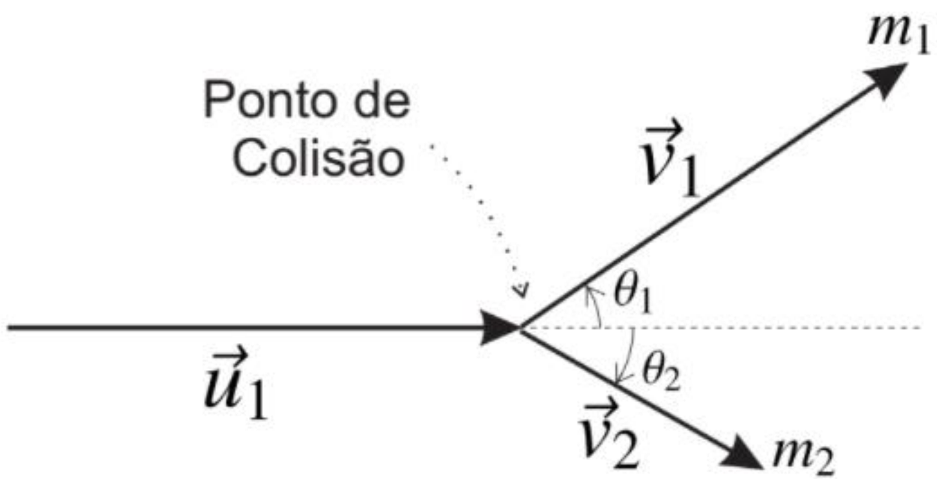
When two particles interact, their movements are determined by the force law that describes their interaction. Such an interaction can be local, such as contact forces when two particles collide, or it can be from some field, such as the Coulomb force between two charged particles.

However, even without knowing the details about this interaction, we can still study some aspects of the collision. Assuming that the interaction is meaningful only when the two particles are sufficiently close, we can investigate what happens at large distances from the collision region. After all, independently from what kind of interaction is involved, it must be such that the total linear momentum of the system is conserved.

We will also suppose that no other particle is created, and that the collision is elastic, in the sense that there is no change of internal state of the particles in this process, that is, the rest mass of each particle is conserved.

For simplicity, consider that in the *lab's frame*, where the things will be measured, one of the particles is initially at rest (we will call it *particle 2*). The other particle (*particle 1*) travels in its direction, until they collide and are eventually scattered.

The image below shows the situation and the basic parameters that describe it.



Particle 1 has rest mass m_1 and speed u_1 and collides with particle 2, of mass m_2, that is stationary in the lab's frame (u_2=0). They leave with velocities *v_1* and *v_2*, forming angles *θ_1* and *θ_2* with the direction of u_1.

This problem is divided into 2 parts: one where we solve the problem classically (1.1) and other where we treat it relativistically (1.2).

1.1 Classical Scattering

a) Show that, in the classical context, the definition of elastic collision mentioned above translates to conservation of energy for the system.

b) It isn't hard to see that this problem can be solved by a much simpler way in the center of mass frame. Make a diagram (like the one in the image above) for the collision as seen by this frame. Show how the conservation laws were used to make this diagram. If you find it convenient, solve questions c) and d) before this one.

c) Calculate the velocities u_1' and u_2' of the particles before the collision, as well as the velocities v_1' and v_2' after the collision, in the center of mass' frame.

d) Suppose that particle 1 is scattered forming an angle φ relative to its initial velocity, in the center of mass reference frame. Is there any restriction for this angle? By what angle will particle 2 be scattered in this frame?

e) Relate the angle φ with the scattering angles in the lab frame, that is, θ_1 and θ_2. Give your answers in terms only of φ, m_1 and m_2.

f) If particle 1 is scattered by a certain angle θ_1, in the lab's frame, then how many solutions for φ exist? Note that there is no need to solve the equation, simply tell how many solutions there are.

g) What is the largest possible scattering angle $\theta_1^{(max)}$ for particle 1 in the lab's frame, in the case m_1>m_2? The answer may depend only on the masses.

h) Discuss briefly the limits m_1>>m_2 and m_1<<m_2.

i) Show that if m_1=m_2 the particles will always be scattered in such a way that θ_1+θ_2=90deg.

1.2 Relativistic Scattering

j) Consider a system of particles of rest mass m_i moving arbitrarily with velocities $\vec{u}_i$, with total linear momentum $\vec{p} = \sum_i \gamma_i m_i \vec{u}_i$, with $\gamma_i = 1/\sqrt{1 - v_i^2/c^2}$, and total energy $E = \sum_i \gamma_i m_i c^2$, for a given reference frame. Show that there always exist a frame (*center of moment*) for which the total linear momentum of the system is zero and the velocity $\vec{u}_{CM}$ of this frame, relative to the original frame, is given by

$$\frac{\vec{u}_{CM}}{c} = \frac{\vec{p}c}{E}$$

Note that this is analogous to the classical center of mass velocity, but with the rest masses substituted by the relativistic ones (that is, $m \to \gamma m$).

k) A particle moves straight with velocity $\vec{u}$ forming and angle α with the x axis of a frame S. A second observer S', that moves with velocity $v\hat{x}$ relative to S, sees the particle forming an angle α' with the x axis. Show that

$$\tan\alpha' = \frac{\sin\alpha}{\gamma(v)(\cos\alpha - v/u)}$$

l) Consider the questions b), c) and d) of part 1.1, but with "center of mass" substituted by "center of moment." What corrections must be made in the answers?

m) Solve question e) of part 1.1 again, for the center of moment. Don't worry about the form of the answer, just make sure that everything that appears in it depends only on the parameters given in the problem, such as the velocities in the lab's frame and the rest masses (and, naturally, the angle φ).

n) Suppose now that m_1=m_2. Show that the angle formed between the scattered particles, in the lab's frame, is always less than π/2rad. Verify that it tends to π/2 in the low velocity limits.

o) Now, let us suppose that a particle is created in the collision. For example, consider a proton (of rest mass M) hitting another proton, at rest. From the collision emerge these 2 protons and also a pion, of rest mass m. Such reaction can be written as $p + p \to p + p + \pi^0$. Assume that the electric interaction between the protons can be neglected. What is the minimum velocity v_0 that the incident proton must have so that this collision is possible?

p) How would you define the efficiency of such a reaction? With your definition, calculate the efficiency of the process above. Note that this question is open-ended, because you need to justify your definition.

2. Black Hole Engine

It is possible that two black holes unite to form another bigger black hole, in a process that increases the entropy of the universe. It is also possible that they are utilized as two thermal reservoirs from where it is possible to extract work. In this problem, we will investigate and isentropic process (at constant entropy) where two black holes unite forming another one, in a process where a certain amount of work can be extracted.

2.1 Thermal Reservoir

The entropy function proposed by Bekenstein for a black home is a function that depends on the square of the hole's mass

$$S_{BN} = 4\pi k \left(\frac{M}{m_p}\right)^2$$

where m_p is a basic constant of nature called *Planck mass*. It can be obtained as a function of **only** the basic constants $\hbar$ (reduced Planck's constant), c (speed of light in vacuum) and G (Newton's constant of gravitation), and of no non-dimensional constants.

a) Determine m_p and its numerical value.

Consider a model in which a black hole is situated inside a box filled by electromagnetic radiation. The walls of the box are perfectly reflective, isolating the external world from the internal world, thus conserving the energy inside. Consider that the black hole is in equilibrium with the radiation.

The total energy of the system is

$$E_{tot} = Mc^2 + aVT^4$$

where the first term represents the energy of the black hole and the second one is the energy due to radiation, with

$$a = \frac{\pi^2 k^4}{15 c^3 \hbar^3}$$

V is the volume of the box and T is the temperature of the radiation. In the above equation, k is Boltzmann's constant.

b) Find the temperature T_{BN} of the black hole and its Schwarzschild radius $R_S = 2GM/c^2$, express R_S as a function of T_{BN}.

The total entropy of the system can be written as $S_{tot} = S_{BN} + S_{rad}$, where S_BN is the entropy of the black hole and S_rad is the entropy of the radiation field. This last one can be written as

$$S_{rad} = \frac{4}{3} a V T^3$$

c) Write the total entropy of the system radiation field + black hole as a function of its mass M and the total mass of the system $M_{tot} = E_{tot}/c^2$, and basic constants.

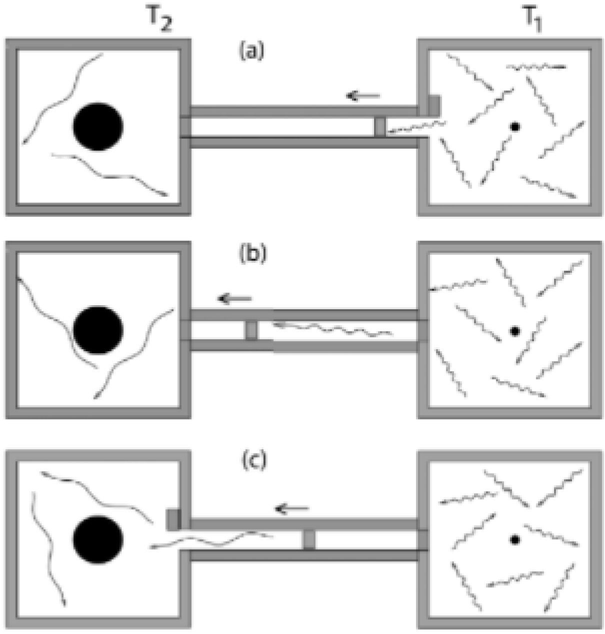
The equilibrium condition for the existence of the black hole in this system is

$$\left(\frac{\partial S_{tot}}{\partial M}\right)_{V,E_{tot}} = 0$$

d) Determine the equation that relates the mass M with the volume V in the equilibrium condition. What must be the condition on the volume V so that this condition can be satisfied, i.e, the box can be very large (small)?

2.2 Carnot Process

Let us consider the Carnot cycle performed by the system illustrated in the figure below, where the work substance used is radiation. The system is composed of two black holes, one big and one small, immersed in radiation fields cold and hot, respectively.



The process consists of 4 stages:

1. The piston is opened in the hot reservoir and moves to increase the volume of the work substance up until V_a.

2. The cylinder is isolated and the radiation expands adiabatically.

3. The cylinder is opened in the cold reservoir and radiation is exchanged.

4. The piston travels through the empty cylinder back to its initial position.

e) Considering that the cylinder that unites the two boxes has negligible volume and that the thermal radiation pressure is given by

$$P_{rad} = \frac{1}{3}aT^4,$$

draw the graph of the Carnot process performed by the work substance in a PV diagram (pressure X volume) and in an ST diagram (entropy X temperature).

f) Determine the total work performed in each stage of the cycle and the exchanged heat.

g) Find the efficiency of the process.

2.3 Energy Extraction

In the last part only one cycle was considered, where the properties of each reservoir practically don't change. However, since in each cycle there is a change of energy in each reservoir, their properties change with time.

h) Determine the relationship between the exchanged heat and the loss (or gain) of mass in the processes 1 and 3 of the cycle described in the last part.

i) Determine the relationship between the masses obtained in the last question and the temperatures of each reservoir.

j) Show that the masses M_1 and M_2 of the reservoir are related with their initial masses M_10 and M_20 through the relationship

$$M_1^2 + M_2^2 = M_{10}^2 + M_{20}^2$$

k) Show that, during the process, the hot reservoir loses mass and becomes hotter while the cold reservoir gains mass and becomes colder, although this is counterintuitive.

l) At the end of the process, when one of the black holes vanishes, what will be the total amount of work extracted from the system?

3. White Dwarf

The Pauli Exclusion Principle is responsible for many properties of matter, both those that we are familiar with, such as the toughness of solid materials, and somewhat stranger things, such as the behavior of white dwarfs, that are very dense stars. In this problem you are challenged to obtain some important results in the study of these stars.

Matter in white dwarfs consists basically of electrons and atomic nuclei, that are basically carbon and oxygen. Because of the fact that they are neutral, the amount of protons and electrons is the same. Also, the number of protons and neutrons is also the same, due to the composition of the stars.

In this problem we will investigate the equilibrium of white dwarfs due to both gravitational interaction and *statistical repulsion* suffered by the *fermions* (electrons, protons and neutrons) that compose the star.

3.1 Kinetic Energy of the Star

In this problem we will consider that the particles that compose the star are non-relativistic.

a) For a given particle of mass m (neutron, proton or electron), write its kinetic energy ε as a function of its mass, wave number k=(2π)/λ and the necessary physical constants.

Now consider, as the model for the star, an infinite box, i.e. a cubic box of side L from where the particles cannot escape. This model is analogous to a rope tied in its ends.

In this model, each particle can only have discrete values of wave number k. The wave number is said *quantized*, in the x, y and z directions.

b) Determine the *quantization condition* of k_x, k_y and k_z in terms of quantum numbers n_x, n_y and n_z for each of the directions. Explain what these numbers mean.

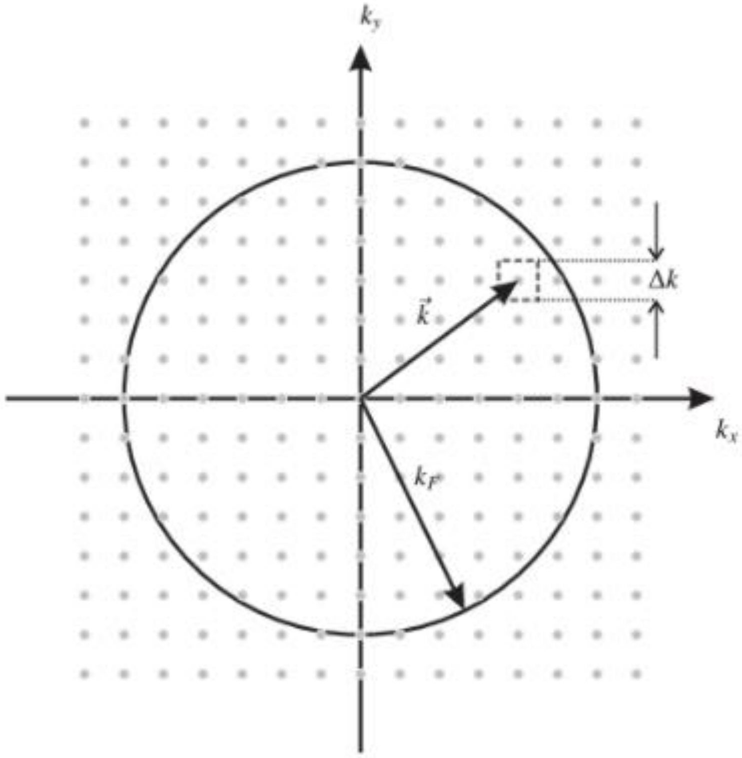
Since the electron is the particle with the smallest mass, it is the particle that contributes the most to the total kinetic energy of the system. Let m_e be the electron mass and m_p the mass of protons and neutrons, considered identical in this problem.

c) Considering that the total mass of the star is M and the mass of the electrons is much larger than that of protons (and neutrons), determine the number of electrons N contained in the star.

d) Determine the smallest difference Δk_i between the possible values of k_i (i=x, y, z).

It is possible to assign to each electron a cube of edge Δk_i, calculated in the previous question, in the wave number space. This means that an electron with wave number $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ occupies a cube of sides Δk_i in the position $\vec{k}$, as shown in the figure below (the figure shows the particular case of 2 dimensions (2D))

The figure also shows the *Fermi's moment* k_F of the system, that is the moment of the highest energy electron.



In this figure, the square indicates the region that occupies two electrons, due to the Pauli exclusion principle.

e) Supposing that all the cubes are filled, starting from 0 up until the Fermi's moment k_F with a max amount of 2 electrons per cube, find the value of the Fermi's moment k_F of the system in terms of the electron density $n = N/L^3 = N/V$ of the system. Consider that the length L is very large. Consider that only positive values of $k_i \ (i = x, y, z)$ are allowed.

f) Find the total kinetic energy of the electrons in the system as a function of the Fermi's moment and the number of electrons in the star.

g) Using the result obtained in the previous question, write the total kinetic energy of the components of the star as a function of its total mass M and radius R. Hint: Use the same result obtained previously, but substitute the volume V by the volume of the star, now considered spherical.

3.2 Potential energy

Now consider that the mass of the star is due entirely to the protons and neutrons. In this case, electrons don't contribute to the gravitational attraction mechanism that keeps the star *alive*.

h) Find the total gravitational potential energy of the star as a function of its mass M and radius R.

i) Sketch the graph of the total energy of the star as a function of the radius R.

j) Discuss which mechanism (gravity or statistical repulsion) is the most important, in the case where the radius R is small and in the case where it is big.

k) Find the stable equilibrium radius of the star, r_0.

Now, suppose that, somehow, it is possible to slightly compress all the mass of the star and shrink its radius to $r_0-\Delta r$, with $\Delta r/r_0 \ll 1$, keeping the density uniform.

l) What will happen with the star? Will it collapse (i.e. the radius R will continuously shrink)? Will the radius vibrate harmonically? With what frequency? Discuss.