Chinese Physics Olympiad 2018 Finals Theoretical Exam Solution

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Note. All numbered equations are marking points (but I am not privy to how much each of them are worth), in that if the marker sees the numbered equation on your answer script (or a similar expression), they will award points accordingly. You may wish to use these as a gauge for how many points you will obtain if you participate in the actual CPhO.

Problem 1 (35 points)**.** Refer to Figure 1.1. A solid hemisphere of radius R and mass M lies at rest upon a smooth tabletop. A smaller solid sphere of uniform density, mass m, and radius r rests upon the apex of the hemisphere. At some instant, the sphere is given a small perturbation and begins to move along the surface of the hemisphere. In the course of the sphere's motion, its position with respect to the hemisphere is described by the angle θ, where θ is the angle between the vertical and the line joining the centres of each body. We are given the moment of inertia of the sphere $\frac{2}{5}mr^2$ about its axis of symmetry, the coefficient of *kinetic* friction μ between the sphere and hemisphere, the assumption that the maximum static friction is equal to the kinetic friction, and the gravitational acceleration g.

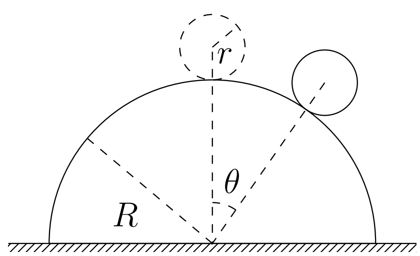

Figure 1.1: A sphere rolling down a hemisphere.

(1) (15 points). The sphere rolls without slipping for a while after it begins to move. When $\theta = \theta_1$, find the magnitudes of the *hemisphere's* velocity $V_M(\theta_1)$ and its acceleration $a_M(\theta_1)$ during this motion.

(2) (15 points). The sphere begins to slip when $\theta = \theta_2$. Find a condition, involving $V_M(\theta_2)$ and $a_M(\theta_2)$, satisfied by θ_2.

(3) (5 points). The sphere loses contact with the hemisphere when $\theta = \theta_3$. Find the speed of the centre of mass of the sphere relative to the hemisphere.

Solution 1:

(1) Since no external forces having nonzero horizontal components act on the bodies, the horizontal component of the momentum of the system is conserved:

$$-MV_M + m[(R+r)\dot{\theta}\cos\theta - V_M] = 0. \tag{1.1}$$

Let ω be the angular speed of the small sphere. Rolling without slipping gives

$$r\omega = (R+r)\dot{\theta}. \tag{1.2}$$

For an explanation of why this works, see e.g. the official solution to the Pan Pearl River Delta Physics Olympiad 2018 Paper 1 Problem 2.

Since no work is done against friction, total mechanical energy is conserved:

$$mg(R+r)(1-\cos\theta) = \frac{1}{2}MV_M^2 + \frac{1}{2}m\left\{\left[(R+r)\dot{\theta}\cos\theta - V_M\right]^2 + \left[(R+r)\dot{\theta}\sin\theta\right]^2\right\} + \frac{1}{2}I\omega^2, \tag{1.3}$$

where $I = \frac{2}{5}mr^2$. Solving (1.1), (1.2), and (1.3) simultaneously, we obtain the speed of the large hemisphere when $\theta = \theta_1$,

$$V_M = \sqrt{\frac{10m^2(R+r)g(1-\cos\theta_1)\cos^2\theta_1}{[7(M+m) - 5m\cos^2\theta_1]\,(M+m)}} \tag{1.4}$$

or

$$V_M^2 = \frac{10m^2(R+r)g(1-\cos\theta_1)\cos^2\theta_1}{[7(M+m) - 5m\cos^2\theta_1]\,(M+m)}.$$

Differentiating the squared equation with respect to t, we obtain

$$2V_M a_M = \frac{10mg(-2\cos\theta + 3\cos^2\theta)[7(M+m) - 5m\cos^2\theta] - 100m^2g(1-\cos\theta)\cos^3\theta}{[7(M+m) - 5m\cos^2\theta]^2} \cdot \frac{m(R+r)\sin\theta\cdot\dot{\theta}}{M+m}.$$

From (1.1) we know that

$$\frac{m(R+r)\dot{\theta}}{M+m} = \frac{V_M}{\cos\theta},$$

so we obtain the acceleration of the hemisphere when $\theta = \theta_1$,

$$a_M(\theta_1) = -\frac{5mg\sin\theta_1\,[14(M+m) - 21(M+m)\cos\theta_1 + 5m\cos^3\theta_1]}{[7(M+m) - 5m\cos^2\theta_1]^2}. \tag{1.5}$$

[*Alternative.* We could also set up equations using Newton's second law and solve from there; see part (2).]

(2) Let N and f be the normal force and friction on the sphere when $\theta \le \theta_2$. By Newton's second law, we obtain

$$N\sin\theta - f\cos\theta = Ma_M. \tag{1.6}$$

Consider the motion of the small sphere in the rest frame of the large hemisphere. We have

$$mg\cos\theta - N - ma_M\sin\theta = m\frac{v_C^2}{R+r}, \tag{1.7}$$

$$mg\sin\theta + ma_M\cos\theta - f = m\frac{\mathrm{d}v_C}{\mathrm{d}t}, \tag{1.8}$$

where v_C is the speed of the small sphere in the rest frame of the large hemisphere,

$$v_C = r\omega = (R+r)\dot{\theta} = \frac{M+m}{m\cos\theta}V_M, \tag{1.9}$$

in which we have used (1.1). Considering the rotation of the small sphere in its rest frame, we have

$$fr = I\frac{\mathrm{d}\omega}{\mathrm{d}t}. \tag{1.10}$$

Solving (1.6), (1.7), (1.8), (1.9), and (1.10) simultaneously, we obtain

$$f = \frac{2m}{7}(g\sin\theta + a_M\cos\theta), \tag{1.11}$$

$$N = mg\cos\theta - ma_M\sin\theta - m\frac{v_C^2}{R+r}. \tag{1.12}$$

Rolling without slipping requires $f \le \mu N$. Equality is achieved when $\theta = \theta_2$, past which the small sphere begins to slip relative to the large hemisphere. Substituting (1.11) and (1.12) into the equation $f = \mu N$, we obtain

$$\frac{2m}{7}[g\sin\theta_2 + a_M(\theta_2)\cos\theta_2] = \mu\left[mg\cos\theta_2 - ma_M(\theta_2)\sin\theta_2 - m\frac{v_C^2(\theta_2)}{R+r}\right].$$

Substituting (1.9) into the above equation yields

$$\frac{2}{7}g\sin\theta_2 - \mu g\cos\theta_2 + a_M(\theta_2)\left(\frac{2}{7}\cos\theta_2 + \mu\sin\theta_2\right) + \frac{\mu(M+m)^2V_M^2(\theta_2)}{(R+r)m^2\cos^2\theta_2} = 0, \tag{1.13}$$

where the expressions for $V_M(\theta_2)$ and $a_M(\theta_2)$ are given by (1.4) and (1.5) respectively.

(3) When the sphere loses contact with the sphere, $N = 0$, at which point the acceleration of the large hemisphere is zero. Therefore, at this moment the speed of the small sphere relative to the large hemisphere ($v_m(\theta_3)$) satisfies the equation

$$mg\cos\theta_3 = m\frac{v_C'^2}{R+r}. \tag{1.14}$$

So we finally have

$$v_C' = \sqrt{(R+r)g\cos\theta_3}. \tag{1.15}$$

Problem 2 (35 points)**.** Both plates (1 and 2) of a parallel-plate capacitor have area S, are fixed horizontally with separation d, and are connected to the circuit shown in Figure 2.1, where U is the emf generated by the power source. An uncharged conducting plate (3) of mass m and having the same dimensions as plates 1 and 2 is placed atop plate 2 and contacts it well. The whole setup is placed inside a vacuum chamber with vacuum permittivity ε_0. When the switch K is closed, plate 3 colliides with plates 1 and 2 in an alternating fashion and undergoes reciprocating motion. We make the following assumptions: the electric field between 1 and 2 is uniform; the resistance of the wires and the internal resistance of the cell are small, so the characteristic charging and discharging times can be neglected; when plate 3 makes contact with plates 1 or 2, the free charge within the contacting plates reaches equilibrium instantly; and all collisions are inelastic. The gravitational acceleration is g.

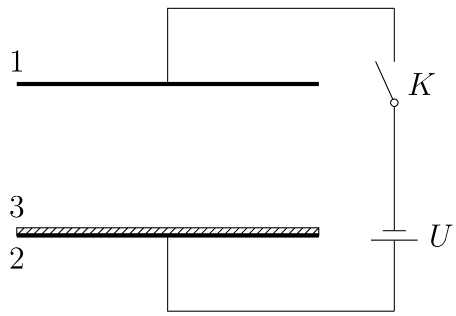

Figure 2.1: A bouncing metal plate.

(1) (17 points). Find the minimum possible value of U.

(2) (18 points). Find the period of the reciprocating motion plate 3 is undergoing.

Note. The integral

$$\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx}} = \frac{1}{\sqrt{a}} \ln\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx}\right) + C$$

is given, where $a > 0$ and C is a constant of integration.

Solution 2:

(1) Just before plate 3 leaves plate 2, the charge on plate 3 is given by

$$Q = C_0 U = \frac{\varepsilon_0 S}{d} U.$$

Suppose that after plate 3 leaves plate 2, the charge on each surface is as shown in Figure 2.2. From charge conservation we obtain

$$\sigma_1 - \sigma_2 = \sigma = \frac{Q}{S} = \frac{\varepsilon_0}{d} U. \tag{2.1}$$

Let E_1 and E_2 be the electric field in between plates 1 and 3 and plates 3 and 2 respectively. We have $E_1 = \sigma_1/\varepsilon_0$ and $E_2 = \sigma_2/\varepsilon_0$. Substituting these into (2.1) and cancelling yields

$$E_1 - E_2 = \frac{U}{d}. \tag{2.2}$$

Since the potential across the capacitors in series is U, we also have

$$E_2 x - E_1(d-x) = U. \tag{2.3}$$

Solving (2.2) and (2.3) simultaneously, we obtain

$$E_1 = \frac{U}{d}\frac{d+x}{d}, \tag{2.4}$$
$$E_2 = \frac{U}{d}\frac{x}{d}. \tag{2.5}$$

Note also that $\sigma_1 = \varepsilon_0 E_1$ and $\sigma_2 = \varepsilon_0 E_2$. Taking upwards as positive hereafter, the electric force acting on plate 3 is given by

$$\begin{aligned} F_e &= -\sigma_2 S \cdot \frac{E_2}{2} + \sigma_1 S \cdot \frac{E_1}{2} = \frac{\varepsilon_0 S}{2}(E_1^2 - E_2^2) = \varepsilon_0 S(E_1 - E_2)\left(\frac{E_1+E_2}{2}\right) \\ &= \varepsilon_0 S \frac{U}{d}\left(\frac{E_1+E_2}{2}\right) = \varepsilon_0 S \frac{U}{d}\cdot\frac{U}{d}\left(\frac{2x+d}{2d}\right) = \frac{\varepsilon_0 S U^3}{2d^3}(2x+d). \end{aligned} \tag{2.6}$$

Thus, the net force acting on plate 3 is given by

$$F_{\text{net}} = F_e - mg = \frac{\varepsilon_0 S U^3}{2d^3}(2x+d) - mg = \left(\frac{\varepsilon_0 S U^3}{2d^2} - mg\right) + \frac{\varepsilon_0 S U^3}{d^3}x.$$

Hence we obtain the acceleration of plate 3 at the position indicated in Figure 2.2,

$$a = \frac{F_{\text{net}}}{m} = \left(\frac{\varepsilon_0 S U^3}{2md^2} - g\right) + \frac{\varepsilon_0 S U^3}{md^3}. \tag{2.7}$$

We need $\varepsilon_0 S U^3/2md^2 \geq g$ in order for plate 3 to rise. The acceleration thereafter is always upwards, so plate 3 is guaranteed to arrive at plate 1. Thus

$$U_{\min} = \sqrt{\frac{2md^2 g}{\varepsilon_0 S}}. \tag{2.8}$$

(2) From (2.7) we know that $a = a_0 + Bx$, where $a_0 = \varepsilon_0 S U^3/2md^2 - g$ and $B = \varepsilon_0 S U^3/md^3$. Thus

$$\begin{aligned} a = v\frac{\mathrm{d}v}{\mathrm{d}x} &= a_0 + Bx \\ v\,\mathrm{d}v &= (a_0 + Bx)\,\mathrm{d}x \\ \int_0^v v'\,\mathrm{d}v' &= \int_0^x (a_0 + Bx')\,\mathrm{d}x' \\ \frac{1}{2}v^2 &= a_0 x + \frac{1}{2}Bx^2 \\ v &= \sqrt{2a_0 x + Bx^2}, \end{aligned}$$

where $v(x)$ is the speed of plate 3 in Figure 2.2. Thus we have

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{2a_0 x + Bx^2}}. \tag{2.9}$$

Integrating on both sides, we obtain the time taken for plate 3 to travel from plate 2 to plate 1 (where $x = d$),

$$\begin{aligned} t_1 &= \int_0^{t_1} \mathrm{d}t = \int_0^{d} \frac{\mathrm{d}x}{\sqrt{2a_0 x + Bx^2}} \\ &= \sqrt{\frac{1}{B}} \ln\left[\frac{(3Bd - 2g) + \sqrt{8Bd(Bd-g)}}{Bd - 2g}\right] \\ &= \frac{d}{U}\sqrt{\frac{md}{\varepsilon_0 S}} \ln\left[\frac{(3\varepsilon_0 SU^2 - 2mgd^2) + 2U\sqrt{2\varepsilon_0 S(\varepsilon_0 SU^2 - mgd^2)}}{\varepsilon_0 SU^2 - 2mgd^2}\right]. \end{aligned} \tag{2.10}$$

Now we consider the subsequent fall of plate 3. When plate 3 collides with plate 1, its speed becomes zero, and it begins falling due to both the effects of gravity and the electric field until it collides inelastically with plate 2, whereupon the process repeats. The computation is similar to the above (but with the E's and σ's primed, as shown in Figure 2.3) and includes marking points (2.11)–(2.18) (roughly corresponding to (2.1)–(2.7) and (2.9) respectively), which are omitted in the interests of the translator's sanity and the health of his poor fingers. After a series of laborious steps (read: bashing) and many tears, we finally obtain the time taken for plate 3 to travel from plate 1 back to plate 2,

$$t_2 = \frac{d}{U}\sqrt{\frac{md}{\varepsilon_0 S}} \ln\left[\frac{(3\varepsilon_0 SU^2 + 2mgd^2) + 2U\sqrt{\varepsilon_0 S(2\varepsilon_0 SU^2 + mgd^2)}}{\varepsilon_0 SU^2 + 2mgd^2}\right]. \tag{2.19}$$

Hence we finally obtain

$$\begin{aligned} T = t_1 + t_2 = \frac{d}{U}\sqrt{\frac{md}{\varepsilon_0 S}} \Bigg\{ &\ln\left[\frac{(3\varepsilon_0 SU^2 - 2mgd^2) + 2U\sqrt{2\varepsilon_0 S(\varepsilon_0 SU^2 - mgd^2)}}{\varepsilon_0 SU^2 - 2mgd^2}\right] \\ &+ \ln\left[\frac{(3\varepsilon_0 SU^2 + 2mgd^2) + 2U\sqrt{\varepsilon_0 S(2\varepsilon_0 SU^2 + mgd^2)}}{\varepsilon_0 SU^2 + 2mgd^2}\right]\Bigg\}. \end{aligned} \tag{2.20}$$

(Phew!)

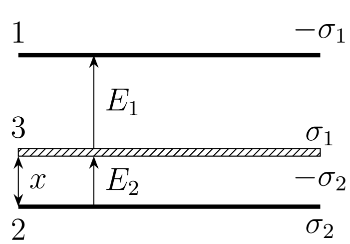

Figure 2.2: The electric field as plate 3 rises.

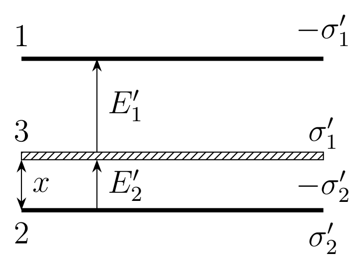

Figure 2.3: The electric field as plate 3 falls.

Problem 3 (35 points)**.** Refer to Figure 3.1. An inextensible, massive string of uniform linear density λ is threaded through a disc-shaped, fixed pulley of radius R, whose axle is a distance L from the floor. The system is initially at rest. When $t = 0$, the pulley acquires a constant angular speed ω (which is maintained throughout) in the anticlockwise direction and causes the string to move as well. The coefficient of kinetic friction between the pulley and the string is μ. The suspended parts of the string are vertical throughout the motion, the ends of the string never leave the floor, and the piles of string resting on the floor are concentrated at two points. We are given the gravitational acceleration g. Denote the tension at the points on the left and right hand sides of the pulley, where the string is tangent to the pulley, by T_1 and T_2 respectively.

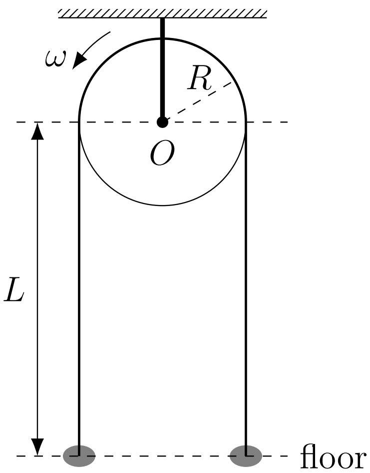

Figure 3.1: A massive string threaded through a rotating pulley. The grey blobs indicate piles of string.

(1) (20 points). Write down a system of dynamical equations for any short length of string at all possible locations on the string, for the time before the speed of the string reaches a maximum. Consider three cases: the two suspended parts and the part threaded through the pulley.

(2) (15 points). Find the maximum possible speed of the string.

Note. The identities

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha y = e^{-\alpha x}\frac{\mathrm{d}(ye^{\alpha x})}{\mathrm{d}x},$$

$$\int e^{\alpha x}\cos x\,\mathrm{d}x = \frac{e^{\alpha x}}{1+\alpha^2}(\alpha\cos x + \sin x) + C_1, \quad \text{and}$$

$$\int e^{\alpha x}\sin x\,\mathrm{d}x = \frac{e^{\alpha x}}{1+\alpha^2}(\alpha\sin x + \cos x) + C_2$$

are given, where C_1 and C_2 are constants of integration.

Solution 3:

(1) We first treat the vertical part on the left, taking downwards as positive. Let v be the speed of the string. In time $\mathrm{d}t$, the pulley releases a piece of string $\mathrm{d}x = v\,\mathrm{d}t$ long, and simultaneously another piece of string of the same length is added to the pile, so the additional momentum from this process is zero. Thus we obtain the change in momentum per

unit time, $\lambda L\,\mathrm{d}v$. Now the deposited length of string does not contribute any force on the suspended length, so the net force on it is $\lambda Lg - T_1$, i.e. the impulse on it is $(\lambda Lg - T_1)\,\mathrm{d}t$. We thus obtain

$$-T_1 + \lambda Lg = \lambda L\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{3.1}$$

We now consider the vertical part on the right, but we take *upwards* as positive. We first consider a small length of string $\mathrm{d}x = v\,\mathrm{d}t$ lifted from the ground. Its momentum instantaneously changes from 0 to $\lambda\,\mathrm{d}x\,v = \lambda v^2\,\mathrm{d}t$. Now the force acting on the suspended segment on the right is given by $T' - \lambda\,\mathrm{d}x\,g$, where T' is the tension of the string near the floor. We thus obtain the equation

$$T' - \lambda\,\mathrm{d}x\,g = \lambda v^2,$$

and after omitting the infinitesimal term obtain

$$T' = \lambda v^2.$$

The change in momentum of the segment on the right is given by $\lambda L\,\mathrm{d}v$, and the net force acting upon it is given by $T_2 - \lambda Lg - T' = T_2 - \lambda Lg - \lambda v^2$. Thus we obtain

$$T_2 - \lambda v^2 - \lambda Lg = \lambda L\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{3.2}$$

Finally, we consider the segment threaded through the pulley, and take anticlockwise as positive. Refer to Figure 3.2. For a length of string subtending an angle of $\Delta\varphi$ at the centre of the pulley, its mass is $\lambda R\Delta\varphi$, the tension at each end are given by $T(\varphi+\Delta\varphi)$ and $T(\varphi)$, the normal force on it is $NR\Delta\varphi$, where N is the normal force per unit length acting on this curved segment of the string. Thus we may set up the dynamical equations along the tangential and normal direction, like so:

$$T(\varphi+\Delta\varphi)\cos\frac{\Delta\varphi}{2} - T(\varphi)\cos\frac{\Delta\varphi}{2} + \mu NR\Delta\varphi - \lambda R\Delta\varphi g\cos\varphi = \lambda R\Delta\varphi\frac{\mathrm{d}v}{\mathrm{d}t},$$
$$T(\varphi+\Delta\varphi)\sin\frac{\Delta\varphi}{2} + T(\varphi)\sin\frac{\Delta\varphi}{2} - NR\Delta\varphi + \lambda R\Delta\varphi g\sin\varphi = \lambda R\Delta\varphi\frac{v^2}{R}.$$

When we take $\Delta\varphi \to 0$, the above system of equations become

$$\frac{\mathrm{d}T}{\mathrm{d}\varphi} + \mu NR - \lambda Rg\cos\varphi = \lambda R\frac{\mathrm{d}v}{\mathrm{d}t}, \tag{3.3}$$
$$T - NR + \lambda Rg\sin\varphi = \lambda R\frac{v^2}{R}. \tag{3.4}$$

Eliminating N from the above system gives

$$\frac{\mathrm{d}T}{\mathrm{d}\varphi} + \mu T - \lambda Rg(\cos\varphi - \mu\sin\varphi) = \lambda R\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \tag{3.5}$$

(2) Since ω can take a range of values, intuitively we find that there are two possible cases: (1) ω is so large that even when v reaches its maximum, there is still relative motion between the pulley and the string; and (2) ω is sufficiently small that when this happens, the string sticks to the pulley such that the maximum speed would be given by $R\omega$.

Alternative 3.1. Note that only T_1 and T_2 remain unknown in (3.1) and (3.2), so we need only obtain a relationship between T_1 and T_2 using (3.5), and we may dispense with $T(\varphi)$ altogether. By the first given identity we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}x} + \mu T = e^{-\mu\varphi}\frac{\mathrm{d}(Te^{\mu\varphi})}{\mathrm{d}x}. \tag{3.6}$$

So (3.5) can be rewritten as

$$\frac{\mathrm{d}(Te^{\mu\varphi})}{\mathrm{d}x} = \lambda Rge^{\mu\varphi}(\cos\varphi - \mu\sin\varphi) + e^{\mu\varphi}\lambda R\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right), \tag{3.7}$$

and after integration from 0 to π on both sides we obtain

$$T_1e^{\mu\pi} - T_2 = \lambda Rg\left(\int_0^\pi e^{\mu\varphi}\cos\varphi\,\mathrm{d}\varphi - \mu\int_0^\pi e^{\mu\varphi}\sin\varphi\,\mathrm{d}\varphi\right) + \lambda R\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right)\int_0^\pi e^{\mu\varphi}\,\mathrm{d}\varphi \tag{3.8}$$

$$T_2 - T_1e^{\mu\pi} = \lambda Rg\frac{2\mu}{1+\mu^2}(e^{\mu\pi}+1) - \frac{\lambda R}{\mu}(e^{\mu\pi}-1)\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \tag{3.9}$$

Alternative 3.2 (Full derivation of $T(\varphi)$). Let

$$T = \bar{T} + C_1\sin\varphi + C_2\cos\varphi + C_3 \tag{3.6}$$

where $\bar{T}$, C_1, C_2, and C_3 are constants to be determined. Substituting this into (3.5), we obtain

$$\begin{aligned} C_1 &= \lambda Rg\frac{1-\mu^2}{1+\mu^2}, \\ C_2 &= \lambda Rg\frac{2\mu}{1+\mu^2}, \\ C_3 &= \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \end{aligned} \tag{3.7}$$

Then the differential equation of interest becomes

$$\frac{\mathrm{d}\bar{T}}{\bar{T}} = -\mu\,\mathrm{d}\varphi$$

and, upon integration, we obtain

$$\bar{T} = \bar{T}_0e^{-\mu\varphi}, \tag{3.8}$$

from which we conclude

$$T = \bar{T}_0e^{-\mu\varphi} + \lambda Rg\frac{1-\mu^2}{1+\mu^2}\sin\varphi + \lambda Rg\frac{2\mu}{1+\mu^2}\cos\varphi + \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right).$$

At the points on both sides of the pulley, where the string is tangent to the pulley, and where $\varphi = 0$ and $\varphi = \pi$, the tension is given by $T_2 = T(0)$ and $T_1 = T(\pi)$, so we obtain

$$\begin{aligned} T_2 &= \bar{T}_0 + \lambda Rg\frac{2\mu}{1+\mu^2} + \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right) \\ T_1 &= \bar{T}_0e^{-\mu\pi} - \lambda Rg\frac{2\mu}{1+\mu^2} + \frac{\lambda R}{\mu}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right), \end{aligned}$$

i.e.

$$T_2 - T_1e^{\mu\pi} = \lambda Rg\frac{2\mu}{1+\mu^2}(e^{\mu\pi}+1) - \frac{\lambda R}{\mu}(e^{\mu\pi}-1)\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \mu\frac{v^2}{R}\right). \tag{3.9}$$

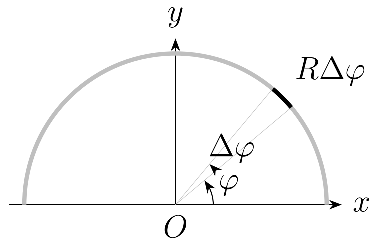

Figure 3.2: An analysis of the segment of string threaded through the pulley.

From (3.1) and (3.2), we also obtain

$$T_2 - T_1 e^{\mu\pi} = \lambda v^2 - \lambda L g(e^{\mu\pi} - 1) + \lambda L(e^{\mu\pi} + 1)\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{3.10}$$

Setting the RHS's of (3.9) and (3.10) equal to each other, we obtain

$$Lg(e^{\mu\pi} - 1) + Rg\frac{2\mu}{1+\mu^2}(e^{\mu\pi} + 1) - e^{\mu\pi}v^2 = \left(L(e^{\mu\pi} + 1) + \frac{R}{\mu}(e^{\mu\pi} - 1)\right)\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{3.11}$$

Now assume that there still remains relative motion between the string and the pulley. When $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$, $v_{\max}$ is achieved, so using (3.11) we may obtain

$$v_{\max}^2 = Lg(1 - e^{-\mu\pi}) + Rg\frac{2\mu}{1+\mu^2}(1 + e^{-\mu\pi}),$$

or

$$v_{\max} = \sqrt{Lg(1 - e^{-\mu\pi}) + Rg\frac{2\mu}{1+\mu^2}(1 + e^{-\mu\pi})}, \tag{3.12}$$

from which we obtain the condition for case (1),

$$R\omega > \sqrt{Lg(1 - e^{-\mu\pi}) + Rg\frac{2\mu}{1+\mu^2}(1 + e^{-\mu\pi})}. \tag{3.13}$$

But if this condition is not satisfied, the string sticks to the pulley, i.e.

$$v_{\max} = R\omega. \tag{3.14}$$

Problem 4 (35 points)**.** Refer to Figure 4.1. A taut string of length L is placed along the x-axis, whose left end is located at the origin. Both ends of the string can be attached to a vibration generator, which drives oscillations in the y-direction. The speed of wave propagation is u.

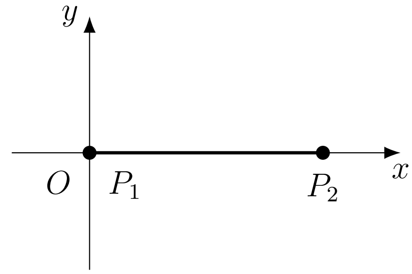

Figure 4.1: A vibrating string.

(1) (22 points). We fix the right end of the string P_2 and connect the left end P_1 to the generator. When the system reaches a steady state, the displacement of the left end is given by $y(x=0,t) = A_0\cos(\omega t)$, where A_0 and ω are the amplitude and angular frequency of the oscillation respectively.

(i) (10 points). We are given that the transverse oscillation attenuates down the string with coefficient $\gamma > 0$. Find the oscillation amplitude everywhere on the string, given that, for a string of infinite length, the equation of a transverse wave travelling and attenuating in the positive x direction is given by $y(x,t) = Ae^{-\gamma x}\cos(\omega t - \omega x/u + \varphi)$, where A and φ are the amplitude and initial phase of the oscillation at $x = 0$ respectively.

(ii) (12 points). We now ignore the effects of attenuation. Find the equation of the standing wave on the string. Find also the positions of the nodes and antinodes of the standing wave.

(2) (13 points). We connect both ends of the string to the generator, such that the displacements of P_1 and P_2 are given by $y(x=0,t) = A_0\cos\omega t$ and $y(x=L,t) = A_0\cos(\omega t + \varphi_0)$ respectively. Ignoring the effects of attenuation, find the equation of the wave everywhere on the string for the cases $\varphi_0 = 0$ and $\varphi_0 = \pi$ respectively, and state the condition for the resonance frequency ω in each case.

Solution 4:

(1) (i) The expression for the wave propagating rightwards on the string is given by

$$y_R(x,t) = A_1 e^{-\gamma x}\cos\left(\omega t - \frac{\omega x}{u} + \varphi_1\right), \tag{4.1}$$

where A_1 and φ_1 are constants to be determined. Since the right end is fixed, the wave experiences a phase change of π as it is reflected off the right end. Thus the expression for the wave propagating leftwards on the string is given by

$$y_L(x,t) = A_1 e^{-2\gamma L + \gamma x}\cos\left(\omega t + \frac{\omega x}{u} - 2\frac{\omega L}{u} + \pi + \varphi_1\right). \tag{4.2}$$

Thus, once the system stabilises, the expression for the transverse oscillation along the string is given by

$$y(x,t) = y_L(x,t) + y_R(x,t) = A(x)\cos[\omega t + \varphi(x)],$$

where

$$A(x) = A_1\sqrt{e^{-2\gamma x} + e^{-4\gamma L+2\gamma x} - 2e^{-2\gamma L}\cos\left(2\frac{\omega}{u}x - 2\frac{\omega L}{u}\right)}. \tag{4.3}$$

Since the amplitude of oscillation at $x = 0$ is given, we have

$$A(x=0) = A_1\sqrt{1 + e^{-4\gamma L} - 2e^{-2\gamma L}\cos\left(2\frac{\omega L}{u}\right)} = A_0,$$

which leads to

$$A_1 = \frac{A_0}{\sqrt{1 + e^{-4\gamma L} - 2e^{-2\gamma L}\cos\left(2\frac{\omega L}{u}\right)}}. \tag{4.4}$$

Hence, the amplitude of the transverse oscillation along the string is given by

$$A(x) = \frac{A_0}{\sqrt{\begin{array}{l}1 + e^{-4\gamma L}\\ \quad - 2e^{-2\gamma L}\cos\left(2\frac{\omega L}{u}\right)\end{array}}}\sqrt{e^{-2\gamma x} + e^{-4\gamma L+2\gamma x} - 2e^{-2\gamma L}\cos\left(2\frac{\omega}{u}x - 2\frac{\omega L}{u}\right)}. \tag{4.5}$$

(ii) If we neglect the effects of attenuation along the string, the expression for standing waves can be written as[a]

$$y(x,t) = A\sin\left(\frac{\omega x}{u} + \varphi\right)\cos(\omega t + \phi). \tag{4.6}$$

The boundary condition at the right end is given by $y(x = L, t) = 0$, so we have

$$\frac{\omega L}{u} + \varphi = m\pi, \quad m = 0,\ 1,\ 2,\ \ldots$$

We may take $m = 0$ so that

$$\varphi = -\frac{\omega L}{u}. \tag{4.7}$$

The other values of m are merely trivial phase shifts of $n\pi$ which may be safely ignored. Using the boundary condition at the left end, $y(x = 0, t) = A_0\cos(\omega t)$, we obtain

$$A\sin\left(-\frac{\omega L}{u}\right)\cos(\omega t + \phi) = A_0\cos\omega t.$$

As the equation above is valid over all t, we conclude that

$$A = -\frac{A_0}{\sin\left(\frac{\omega L}{u}\right)}, \tag{4.8}$$

$$\phi = 0. \tag{4.9}$$

By substituting (4.8) and (4.9) into (4.6), the expression for standing waves becomes

$$y(x,t) = -\frac{A_0}{\sin\left(\frac{\omega L}{u}\right)}\sin\left(\frac{\omega x}{u} - \frac{\omega L}{u}\right)\cos(\omega t). \tag{4.10}$$

From (4.10) we know that the amplitude of the standing wave at position x is given by

$$A(x) = \left| \frac{A_0}{\sin\left(\frac{\omega L}{u}\right)} \sin\left(\frac{\omega x}{u} - \frac{\omega L}{u}\right) \right|.$$

At the nodes where $x = x_n$, $A(x) = 0$, which implies that

$$\sin\left(\frac{\omega x}{u} - \frac{\omega L}{u}\right) = 0,$$

from which we obtain

$$x_n = L - n\pi\frac{u}{\omega}, \tag{4.11}$$

where n is an integer satisfying the inequality

$$0 \le n < \frac{L\omega}{\pi u}. \tag{4.12}$$

At the antinodes where $x = x_a$, $A(x)$ is maximum, so we obtain

$$x_a = L - \pi\frac{u}{2\omega} - l\pi\frac{u}{\omega}, \tag{4.13}$$

where l is an integer satisfying the inequality

$$0 < l < \frac{L\omega}{\pi u}. \tag{4.14}$$

(2) We decompose the expression for the transverse oscillation along the string into the superposition of the expressions arising from two different sets of boundary conditions:

(A) Left end vibrates, right end fixed; and

(B) Left end fixed, right end vibrates.

Let $y_1(x,t)$ be the expression corresponding to scenario (A), and let $y_2(x,t)$ be the expression corresponding to scenario (B). Using the results of (1)(ii), we write down the expression for standing waves in (A):

$$y_1(x,t) = -\frac{A_0}{\sin\left(\frac{\omega L}{u}\right)} \sin\left(\frac{\omega x}{u} - \frac{\omega L}{u}\right)\cos(\omega t);$$

And in (B):

$$y_2(x,t) = \frac{A_0}{\sin\left(\frac{\omega L}{u}\right)} \sin\left(\frac{\omega x}{u}\right)\cos(\omega t + \varphi_0). \tag{4.15}$$

When both ends are vibrating, the corresponding expression is the superposition of the above two expressions, so we may write

$$y(x,t) = y_1(x,t) + y_2(x,t) = \frac{A_0}{\sin\left(\frac{\omega L}{u}\right)} \left[\sin\left(\frac{\omega L}{u} - \frac{\omega x}{u}\right)\cos(\omega t) + \sin\left(\frac{\omega x}{u}\right)\cos(\omega t + \varphi_0)\right].$$

When $\varphi_0 = 0$, we have

$$y(x,t) = \frac{A_0}{\cos\left(\frac{\omega L}{2u}\right)} \cos\left(\frac{\omega L}{2u} - \frac{\omega x}{u}\right) \cos \omega t. \tag{4.16}$$

When resonance occurs, $\cos(\omega L/2u) = 0$, so we obtain

$$\omega = \frac{(2n_1+1)\pi u}{L}, \quad n_1 = 0,\ 1,\ 2,\ 3,\ \ldots \tag{4.17}$$

When $\varphi_0 = \pi$, we have

$$y(x,t) = \frac{A_0}{\sin\left(\frac{\omega L}{2u}\right)} \sin\left(\frac{\omega L}{2u} - \frac{\omega x}{u}\right) \cos \omega t. \tag{4.18}$$

When resonance occurs, $\sin(\omega L/2u) = 0$, so we obtain

$$\omega = \frac{2n_2\pi u}{L}, \quad n_2 = 1,\ 2,\ 3,\ \ldots \tag{4.19}$$

[a] *Translator's note.* In China, ϕ and φ are treated as different symbols. They are not interchangeable.

Problem 5 (35 points). An insulated, thin-walled container of mass M is placed in outer space, far away from any celestial bodies, such that outer space can be modelled as a vacuum. The initial velocity of the container is zero when observed in a certain inertial reference frame. The capacity of the container is V, and it is initially filled with N_0 molecules of a monatomic ideal gas, whose individual mass is m, and whose initial temperature is T_0. When $t = 0$, the container is punctured, and a hole of area S appears on the wall. The container begins to move but does not rotate due to the gas leak. We assume that the hole is small and that the ideal gas remains in thermodynamic equilibrium throughout the process. We are given the Maxwell-Boltzmann distribution function

$$f(v_x) = \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv_x^2}{2kT}\right)$$

for the x-component of the molecular velocity v_x, where k is the Boltzmann constant. Find

(1) (6 points) the number of molecules escaping the container per unit area per unit time, in terms of the molecular number density n and the temperature T of the gas at that instant;

(2) (6 points) the average kinetic energy of each molecule relative to the container, in terms of T;

(3) (15 points) the temperature of the gas at time t; and

(4) (8 points) the speed of the container at time t.

Note. We are given the identities

$$\int_0^\infty x e^{-Ax^2}\,\mathrm{d}x = \frac{1}{2A},$$
$$\int_0^\infty x^2 e^{-Ax^2}\,\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{A^3}}, \quad \text{and}$$
$$\int_0^\infty x^3 e^{-Ax^2}\,\mathrm{d}x = \frac{1}{2A^2}.$$

Solution 5:

(1) WLOG assume that the plane of the hole is orthogonal to the x-axis. In time $\mathrm{d}t$, the number of leaked gas molecules is equal to the number of gas molecules within a prism of volume $v_x S\,\mathrm{d}t$ whose velocity component in the x-direction is within the range v_x to $v_x + \mathrm{d}v_x$. The required number, therefore, is given by $n_{v_x} S v_x\,\mathrm{d}t\,\mathrm{d}v_x$, where n_{v_x} is the number density of molecules with x-velocity v_x, and is related to n by the Maxwell-Boltzmann distribution function:

$$n_{v_x} = nf(v_x) = n\sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv_x^2}{2kT}\right). \tag{5.1}$$

Thus the average total number of leaked gas molecules per unit area per unit time is given by

$$N_{\text{leak}} = \int_0^\infty n_{v_x} v_x\,\mathrm{d}v_x = \sqrt{\frac{m}{2\pi kT}} \int_0^\infty v_x n \exp\left(-\frac{mv_x^2}{2kT}\right)\mathrm{d}v_x = n\sqrt{\frac{kT}{2\pi m}}. \tag{5.2}$$

(2) When the gas temperature is T, the average kinetic energy of the leaked gas molecules is

given by

$$\bar{E}_k = \frac{1}{N_{\text{leak}}} \int_0^\infty \frac{1}{2} m v_x^2 n_{v_x} v_x \, \mathrm{d}v_x + kT, \tag{5.3}$$

in which kT is the contribution of the velocity components in the y- and z-directions (two additional degrees of freedom). Substituting (5.1) and (5.2) into (5.3), we obtain

$$\begin{aligned}\bar{E}_k &= \frac{1}{n\sqrt{\frac{kT}{2\pi m}}} \int_0^\infty \frac{1}{2} mn \sqrt{\frac{m}{2\pi kT}} v_x^3 \exp\left(-\frac{m v_x^2}{2kT}\right) \mathrm{d}v_x + kT \\ &= \frac{1}{\sqrt{\frac{kT}{2\pi m}}} \frac{1}{2} m \sqrt{\frac{m}{2\pi kT}} \frac{1}{2(m/2kT)^2} + kT = 2kT.\end{aligned} \tag{5.4}$$

(3) As the hole is orthogonal to the x-direction, the velocity of the container will point along the $-x$ direction as a result of the gas leak in the $+x$ direction, due to symmetry considerations. When the time is t, let N be the number of molecules inside the container, and let u be the speed of the container; and when the time is $t + \mathrm{d}t$, let $N + \mathrm{d}N$ and $u + \mathrm{d}u$ be the corresponding values. Then momentum is conserved throughout the process, so we may write

$$(M + Nm)u = [(M + Nm + m\,\mathrm{d}N)(u + du)] - m\,\mathrm{d}N\,(u - \bar{v}_x), \tag{5.5}$$

where $\bar{v}_x$ is the average x-component of the velocity of leaked gas molecules in time $\mathrm{d}t$; note that $\mathrm{d}N$ is, of course, negative. Simplifying and omitting higher-order terms, we obtain

$$(M + Nm)\,\mathrm{d}u = -m\bar{v}_x\,\mathrm{d}N\,. \tag{5.6}$$

Energy is also conserved throughout the process, so we may also write

$$\begin{aligned}\frac{1}{2}(M + Nm)u^2 + \frac{3}{2}NkT = \frac{1}{2}(M + Nm + \mathrm{d}N\,m)(u + \mathrm{d}u)^2 + \frac{3}{2}(N + \mathrm{d}N)k(T + \mathrm{d}T) \\ - \frac{1}{2}\,\mathrm{d}N\,m\overline{(u - v_x)^2} - \mathrm{d}N\,kT,\end{aligned} \tag{5.7}$$

where the two terms on the LHS are the translational kinetic energy and the internal energy of the container before dtm, whereas the former two terms on the RHS are the translational kinetic energy and internal energy of the container after dt, and the latter two represent the average kinetic energy of the leaked gas molecules. Simplifying and omitting higher-order terms as before, we obtain

$$(M + Nm)u\,\mathrm{d}u + \frac{3}{2}Nk\,\mathrm{d}T + \frac{1}{2}kT\,\mathrm{d}N + mu\bar{v}_x\,\mathrm{d}N - \frac{1}{2}m\overline{v_x^2}\,\mathrm{d}N = 0. \tag{5.8}$$

Substituting (5.6) into (5.8) gives

$$\left(\frac{1}{2}m\overline{v_x^2} - \frac{1}{2}kT\right)\mathrm{d}N = \frac{3}{2}Nk\,\mathrm{d}T\,. \tag{5.9}$$

Using the result of (5.4) we may write

$$\begin{aligned}\frac{1}{2}m\overline{v_x^2} + kT &= 2kT \\ \frac{1}{2}m\overline{v_x^2} &= kT\end{aligned}$$

Substituting this into (5.9), we obtain

$$\frac{1}{2}kT\,\mathrm{d}N = \frac{3}{2}Nk\,\mathrm{d}T\,,$$

which is just

$$\frac{\mathrm{d}N}{N} = 3\frac{\mathrm{d}T}{T}. \tag{5.10}$$

Integrating on both sides gives

$$T = CN^{1/3}, \tag{5.11}$$

where C is an unknown constant. Substituting in the initial conditions gives

$$T_0 = CN_0^{1/3} \qquad C = \frac{T_0}{N_0^{1/3}}. \tag{5.12}$$

Using (5.2), the number of gas molecules leaked in time $\mathrm{d}t$ is given by

$$\mathrm{d}N = -\frac{N}{V}S\sqrt{\frac{kT}{2\pi m}}\,\mathrm{d}t = -\frac{N^{7/6}}{V}S\sqrt{\frac{kC}{2\pi m}}\,\mathrm{d}t\,. \tag{5.13}$$

Integrating on both sides gives

$$N = N_0\left(1+\frac{S}{6V}\sqrt{\frac{kT_0}{2\pi m}}t\right)^{-6}. \tag{5.14}$$

Hence we obtain our desired result,

$$T = CN^{1/3} = T_0\left(1+\frac{S}{6V}\sqrt{\frac{kT_0}{2\pi m}}t\right)^{-2}. \tag{5.15}$$

(4) From (5.6) we obtain

$$\mathrm{d}u = -\frac{m\bar{v}_x\,\mathrm{d}N}{M+Nm}, \tag{5.16}$$

in which $\bar{v}_x$ is the average speed of the leaked gas molecules, which can be written as

$$\begin{aligned}\bar{v}_x &= \frac{1}{N_{\text{leak}}}\int_0^\infty v_x n_{v_x} v_x\,\mathrm{d}v_x = \frac{1}{\sqrt{\frac{kT}{2\pi m}}}\int_0^\infty \sqrt{\frac{m}{2\pi kT}}v_x^2\exp\left(-\frac{mv_x^2}{2kT}\right)\mathrm{d}v_x \\ &= \frac{1}{4}\frac{1}{\sqrt{\frac{kT}{2\pi m}}}\sqrt{\frac{m}{2\pi kT}}\sqrt{\frac{\pi}{(m/2kT)^3}} = \sqrt{\frac{\pi kT}{2m}}.\end{aligned} \tag{5.17}$$

Substituting (5.17) into (5.16) gives

$$u(t) = -\int_{N_0}^{N(t)}\sqrt{\frac{\pi mkT}{2}}\frac{\mathrm{d}N}{M+Nm} = -\sqrt{\frac{\pi mkT_0}{2N_0^{1/3}}}\int_{N_0}^{N(t)}\frac{N^{1/6}\,\mathrm{d}N}{M+Nm} \tag{5.18}$$

When the mass of the container is far larger than the mass of the gas within, i.e. $M \gg N_0 m$, the previous expression can be written approximately as

$$u(t) \approx -\frac{1}{M}\sqrt{\frac{\pi mkT_0}{2N_0^{1/3}}}\int_{N_0}^{N(t)} N^{1/6}\,\mathrm{d}N = -\frac{6}{7M}\sqrt{\frac{\pi mkT_0}{2N_0^{1/3}}}\left(N(t)^{7/6}-N_0^{7/6}\right). \tag{5.19}$$

Substituting (5.14) into (5.19), we obtain our final expression for the speed of the container at time t:

$$u(t) \approx \frac{6N_0}{7M}\sqrt{\frac{\pi m k T_0}{2}}\left(1 - \left(1 + \frac{S}{6V}\sqrt{\frac{kT_0}{2\pi m}}t\right)^{-7}\right). \tag{5.20}$$

Problem 6 (35 points)**.** The refractive index n of an optical medium can either be greater than or less than zero. Media in which $n < 0$ are called negative-index meta-materials (NIMs). When light propagates in a NIM, its optical path length is negative.[1] If we say that the angle of refraction is negative when the incident ray and refracted ray are on the same side of the normal, Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

would still hold, even if the refractive index on either side were negative. Here, n_1 and n_2 can either be positive or negative, and θ_2 is the angle of refraction. We use the convention where $\theta_1 \geq 0$ always.

(1) (10 points). Suppose that a beam of light is incident upon an interface between two materials with different refractive indices. For each possible case as depicted in Figures 6.1 and 6.2, show the path of the rays entering material 2 and the corresponding wavelets on the figures. Hence, show that the generalised Snell's law holds.

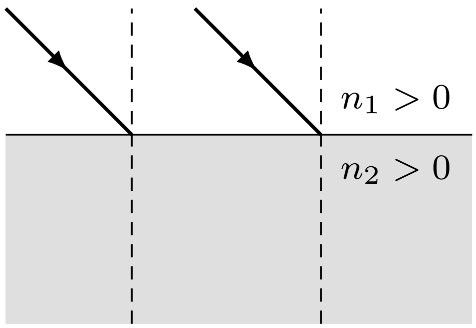

Figure 6.1

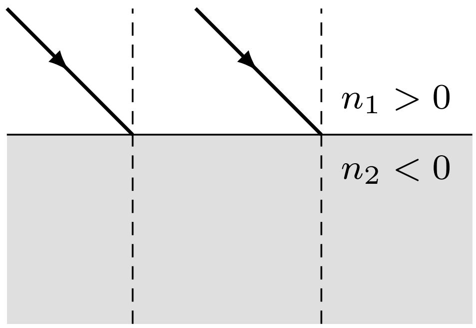

Figure 6.2

(2) (13 points). Refer to Figure 6.3. A spherical surface of radius R and centre C partitions three-dimensional space into an outer region and an inner region, with refractive indices $n_1 > 0$ and $n_2 < 0$ respectively. Considering any optical axis passing through C, we set the origin at O, the intersection of the axis with the interface. Light ray x is incident upon the interface at M and refracts to give ray y, as shown. Let s_1 and s_2 be the object distance and image distance respectively. Under the paraxial approximation, derive the equivalent lens equation (a relationship between s_1, s_2, and the given parameters of the system) and an expression for the magnification of an image. Note down the sign of each quantity in the final results *explicitly*.

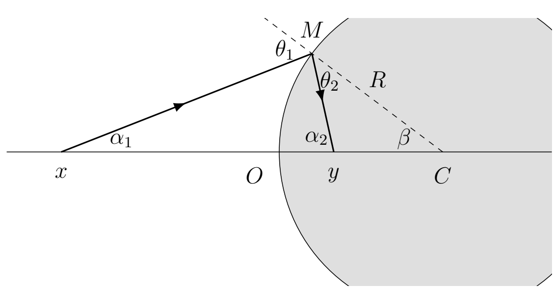

Figure 6.3: A spherical NIM.

[1] When we say that the optical path length of a light ray is negative, we mean that, as the light propagates, its phase change is the opposite of the case where it is propagating in a typical medium. In other words, we can model a light ray travelling in a NIM of some refractive index $n_2 < 0$ by treating it as if it were travelling in the *opposite* direction in a medium of refractive index $-n_2 > 0$.

(3) (12 points). Suppose that medium 1 is air, i.e. $n_1 \approx 1$, and n_2 can either take positive or negative values. We place a thin convex lens of focal length $f = 1.5R$ in front of the spherical interface such that its optical axis passes through C, one focal point is inside the NIM, and the distance between O and the centre of the lens O' is d. A beam of light rays all parallel to the axis is incident upon the lens. For each set of parameters in Table 6.1, obtain the distance between O and the point at which all light rays converge. Also draw a figure illustrating the path of the light rays in case 4.

Case	n_2	d
1	1.5	$0.35R$
2	1.5	$0.85R$
3	-1.5	$0.35R$
4	-1.5	$0.85R$

Table 6.1

Solution 6:

(1) For the case $n_1 > 0$ and $n_2 > 0$, we draw Figure 6.4. Let $\overline{AB}$ be a wavefront, along which the wave has the same phase. When the wave at B reaches B', the front of the wavelet propagating from A is a hemispherical surface of radius r situated within medium 2, whose intersection with the incident plane is a semicircle also of radius r. r is, in turn, determined by

$$n_2 r = n_1 BB'. \tag{6.1}$$

Let line $A'B'$ be the tangent of the line from B' to the wave at A'; then $\overline{A'B'}$ is also a wavefront. By the results of geometry, we have

$$BB' = AB' \sin\theta_1,$$
$$r = AB' \sin\theta_2,$$

from which it is straightforward to derive Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{6.2}$$

We now treat the case $n_1 > 0$ and $n_2 < 0$. Recall that the optical path length of light travelling in medium 2 is negative.

Alternative 6.1. As shown in Figure 6.5a, the wavelet propagating from B' is given by the semicircle of radius r situated within medium 2, and the corresponding optical path length is given by $n_2 r < 0$. Therefore, the wavefront including A is given by the tangent from A to this semicircle, whose point of tangency we name A'. The radius of the semicircle is determined by

$$n_1 BB' + n_2 A'B' = 0. \tag{6.3}$$

By geometry

$$BB' = AB' \sin\theta_1, \qquad B'A' = AB' \sin|\theta_2|,$$

from which we may derive the generalised Snell's law:

$$n_1 AB' \sin\theta_1 = -n_2 AB' \sin|\theta_2| = n_2 AB' \sin\theta_2 \tag{6.4}$$
$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{6.5}$$

Alternative 6.2. As shown in Figure 6.5b, $\overline{AB}$ is a wavefront of the incident light. When the wave at B reaches B', the wavelet propagating from A is a circle of centre A and radius r, corresponding to the optical path length given by $n_2 r < 0$. The radius is determined by

$$|n_2|r = n_1 BB'. \tag{6.3}$$

Since the optical path length is negative, we may as well treat the wavefronts in medium 2 as moving in the opposite direction as compared to the typical case, i.e. away from medium 2 and towards medium 1. Thus the wavelet of interest is the dashed semicircle in Figure 6.5b. If we construct line $B'A'$ tangent to the semicircle at A', then $\overline{B'A'}$ is a wavefront. By geometry

$$BB' = AB' \sin\theta_1, \qquad r = AB' \sin|\theta_2|, \tag{6.4}$$

so we obtain

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{6.5}$$

(2) We take $s_1 > 0$ when the object is outside the NIM and $s_2 > 0$ when the image is inside the NIM; the reverse is true for $s_1 < 0$ and $s_2 < 0$. We take $\theta_1 > 0$ (incident angle) and $\theta_2 < 0$ (refracted angle), and take all other named angles in Figure 6.3 as positive. We notice that

$$\theta_1 = \alpha_1 + \beta, \qquad -\theta_2 = \alpha_2 - \beta. \tag{6.6}$$

Under the paraxial approximation, all these angles are small, so we have

$$\sin\theta_1 = \theta_1, \qquad \sin\theta_2 = \theta_2$$

and

$$\begin{aligned} \tan\alpha_1 = \sin\alpha_1 = \alpha_1 = \frac{h}{s_1}, \\ \tan\alpha_2 = \sin\alpha_2 = \alpha_2 = \frac{h}{s_2}, \\ \tan\beta = \sin\beta = \beta = \frac{h}{R}, \end{aligned} \tag{6.7}$$

where h is the distance of the point of incidence M from the optical axis. Snell's law yields

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$n_1\left(\frac{h}{s_1} + \frac{h}{R}\right) = -n_2\left(\frac{h}{s_2} - \frac{h}{R}\right) \tag{6.8}$$

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}, \tag{6.9}$$

where (6.9) is the required effective lens equation. Note that since $n_2 < 0$ and $n_1 > 0$, the RHS of (6.9) is negative.

To find the magnification, we draw Figure 6.6 and consider object $\overline{Q_1P_1}$ and its image, $\overline{Q_2P_2}$. Now suppose we rotate the parts in blue clockwise about C, bringing points Q_1, O, and Q_2 along with them, such that P lies on the rotated optical axis. Let Q_1', O', and Q_2' be the transformed positions of the previously-named points respectively. Under the paraxial

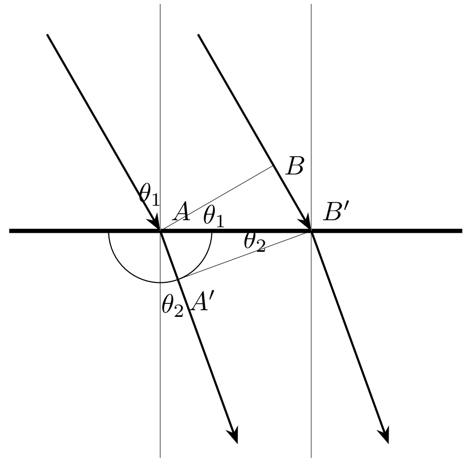

Figure 6.4: Huygens's principle applied to a typical material.

approximation, since the angle through which the blue parts are rotated by is small, we may write

$$Q_1P_1 = Q_1Q_1', \quad Q_2P_2 = Q_2Q_2'. \tag{6.10}$$

Thus by the above and geometry we have

$$Q_1P_1 = s_1\theta_1, \tag{6.11}$$
$$Q_2P_2 = s_2\theta_2, \tag{6.12}$$

which yields our desired magnification,

$$\frac{Q_2P_2}{Q_1P_1} = \frac{-s_2\theta_2}{s_1\theta_1} = -\frac{s_2n_1}{s_1n_2}. \tag{6.13}$$

(3) As shown in Figure 6.7, if the spherical NIM is absent, then the light rays would converge at the focus of the convex lens after passing through it, as shown by the dashed lines. When the NIM is introduced, the original point of convergence becomes the virtual object awaiting transformation by the NIM, so the corresponding object distance is given by

$$s_1 = -(f-d). \tag{6.14}$$

If $n_2 > 0$, the equivalent lens equation is given by (this is in the syllabus of CPhO finals):

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}, \tag{6.15}$$

which is of the same form as (6.9). Solving for s_2 in both cases yields

$$s_2 = \frac{n_2s_1R}{n_2s_1 - n_1(s_1+R)} = \frac{n_2R(f-d)}{(n_2-n_1)(f-d)+n_1R}. \tag{6.16}$$

The required results are shown in Table 6.2. The required diagram is shown in Figure 6.7.

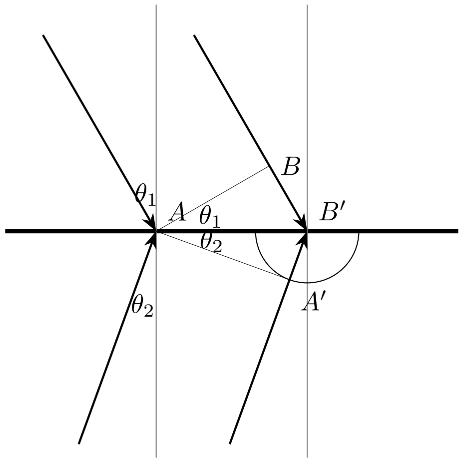

Figure 6.5a: Huygens's principle applied to a NIM.

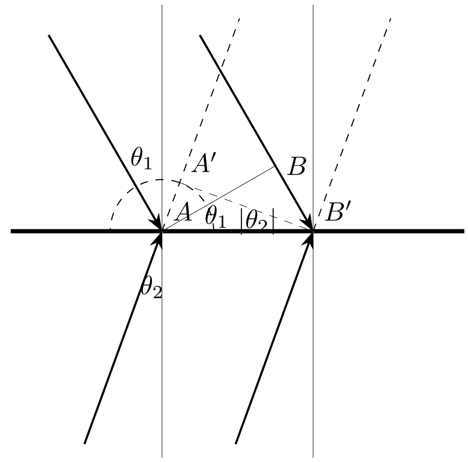

Figure 6.5b: Huygens's principle applied to a NIM—Alternative.

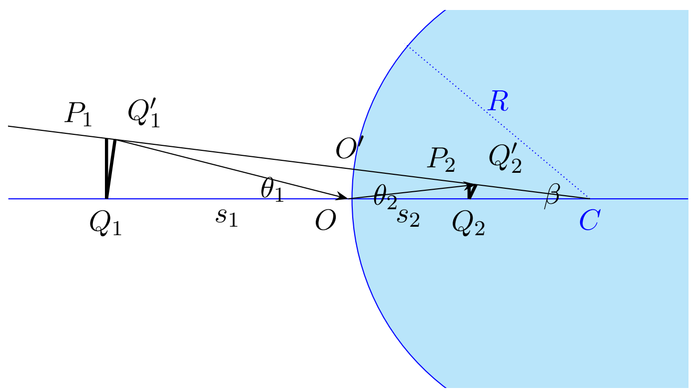

Figure 6.6: Construction for finding the magnification of the spherical NIM. Location of Q_1' and Q_2' not to scale for clarity.

Case	n_2	d	s_2
1	1.5	$0.35R$	$1.10R$
2	1.5	$0.85R$	$0.74R$
3	-1.5	$0.35R$	$0.92R$
4	-1.5	$0.85R$	$1.56R$

Table 6.2

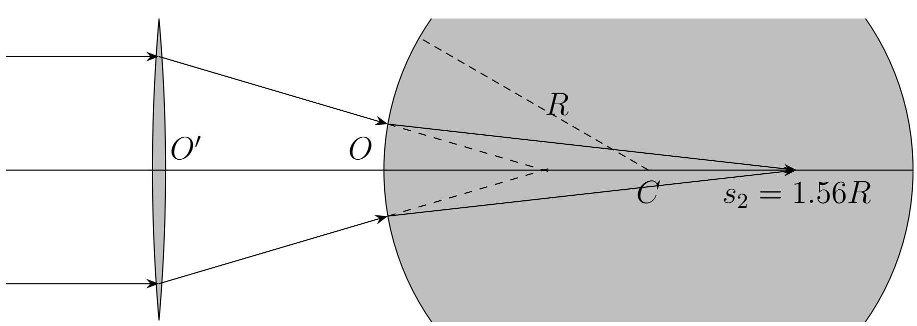

Figure 6.7: Ray diagram for lens-NIM setup.

Problem 7 (35 points)**.** Modelling the physical behaviour of solids, which often have complicated internal structures, can be quite challenging using conventional means alone. If we wish to simplify the problem while still taking the interactions between constituent particles into account, we may use the concept of *quasiparticles*, for which the energy-momentum relation may be different from the one which usually applies to real particles. When external electric or magnetic fields are applied on a solid, the motion of quasiparticles can typically be treated with the methods of classical mechanics.

One type of quasiparticle of effective mass m and carrying charge q exists within some two-dimensional interface-like structure. Its motion is constrained to the xy-plane. Its kinetic energy K can be expressed in terms of the magnitude of its momentum p by the equation

$$K = \frac{p^2}{2m} + \alpha p$$

where α is a positive constant.

(1) (4 points). For a *real* particle of mass m in free motion, its kinetic energy K can be expressed in terms of the magnitude of its momentum p by $K = p^2/2m$. Express its velocity $\mathbf{v}$ in terms of its momentum $\mathbf{p}$ using the work-energy theorem.

(2) (5 points). Using a similar method, express the velocity $\mathbf{v}$ of a *quasiparticle* in terms of its momentum $\mathbf{p}$.

(3) (4 points). Express $v = |\mathbf{v}|$ in terms of K.

(4) (11 points). We now place the two-dimensional interface within a uniform magnetic field of magnitude B and pointing in the $+z$ direction. For a quasiparticle of kinetic energy K, which will undergo uniform circular motion, find the radius of its trajectory, the period of its motion, and the magnitude of its angular momentum.

(5) (11 points). We replace the magnetic field with a uniform electric field of magnitude E and pointing in the $+x$ direction. Note that the component of the quasiparticle's acceleration perpendicular to the electric field may be nonzero. Find the components of the quasiparticle's acceleration a_x and a_y when it moves with speed v and its velocity makes an angle θ with the electric field.

Solution 7:

(1) The work-energy theorem gives us

$$\mathrm{d}K = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \cdot \mathrm{d}\mathbf{r} = \mathbf{v} \cdot \mathrm{d}\mathbf{p}\,. \tag{7.1}$$

Differentiating the given energy-momentum relation for *real particles* on both sides with respect to $\mathbf{p}$ also gives us

$$\mathrm{d}K = \frac{\mathbf{p}}{m} \cdot \mathrm{d}\mathbf{p}\,, \tag{7.2}$$

in which we note that $\mathrm{d}(p^2) = \mathrm{d}(\mathbf{p} \cdot \mathbf{p}) = \mathbf{p} \cdot \mathrm{d}\mathbf{p} + \mathrm{d}\mathbf{p} \cdot \mathbf{p} = 2(\mathbf{p} \cdot \mathrm{d}\mathbf{p})$. Setting the RHS's of (7.1) and (7.2) equal, we have

$$\left(\frac{\mathbf{p}}{m} - \mathbf{v}\right) \cdot \mathrm{d}\mathbf{p} = 0,$$

and combined with the fact that the above is true for all $\mathrm{d}\mathbf{p}$, we confirm that

$$\mathbf{v} = \frac{\mathbf{p}}{m}, \tag{7.3}$$

and that we haven't gone insane (yet).

(2) Differentiating the given energy-momentum relation for *quasiparticles* on both sides with respect to $\mathbf{p}$ gives us

$$\mathrm{d}K = \frac{\mathbf{p}}{m} \cdot \mathrm{d}\mathbf{p} + \alpha\hat{\mathbf{p}} \cdot \mathrm{d}\mathbf{p}\,. \tag{7.4}$$

Again setting the RHS's of (7.1) and (7.4) equal, we have

$$\left(\frac{\mathbf{p}}{m} + \alpha\hat{\mathbf{p}} - \mathbf{v}\right) \cdot \mathrm{d}\mathbf{p} = 0,$$

and by the same argument

$$\mathbf{v} = \frac{\mathbf{p}}{m} + \alpha\hat{\mathbf{p}}. \tag{7.5}$$

(3) Squaring (7.5) on both sides, we obtain

$$v^2 = \frac{p^2}{m^2} + 2\alpha\frac{p}{m} + \alpha^2 = \left(\frac{p}{m} + \alpha\right)^2,$$

so

$$v = \frac{p}{m} + \alpha. \tag{7.6}$$

Making p the subject of the energy-momentum relation for quasiparticles, we have

$$p = -m\alpha + \sqrt{m^2\alpha^2 + 2mK}, \tag{7.7}$$

so substituting (7.7) into (7.6), we obtain

$$v = \sqrt{\alpha^2 + \frac{2K}{m}}. \tag{7.8}$$

(4) In a uniform magnetic field, the quasiparticle undergoes uniform circular motion with speed $v = \omega R$ and rate of change of momentum $|\,\mathrm{d}\mathbf{p}/\mathrm{d}t\,| = \omega p$, so

$$\left|\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right| = \frac{v}{R}p. \tag{7.9}$$

The quasiparticle experiences a Lorentz force related to the above by

$$p\frac{v}{R} = qvB, \tag{7.10}$$

so substituting in (7.7), we obtain the radius of the quasiparticle's trajectory

$$R = \frac{p}{qB} = \frac{-m\alpha + \sqrt{m^2\alpha^2 + 2mK}}{qB}, \tag{7.11}$$

the period of its motion

$$T = \frac{2\pi R}{v} = \frac{2\pi\left(-m\alpha + \sqrt{m^2\alpha^2 + 2mK}\right)}{qB\sqrt{\alpha^2 + \frac{2K}{m}}}, \tag{7.12}$$

and, since the magnitude of its angular momentum is given by

$$L = pR, \tag{7.13}$$

we may substitute (7.7) and (7.11) into (7.13) to obtain

$$L = \frac{2m^2\alpha^2 + 2mK - 2m\alpha\sqrt{m^2\alpha^2 + 2mK}}{qB}. \tag{7.14}$$

(5) (7.5) gives us

$$\mathbf{p} = m\mathbf{v} - m\alpha\hat{\mathbf{v}}, \tag{7.15}$$

keeping in mind that $\hat{\mathbf{p}} = \mathbf{p}/p$ and similar. Newton's second law yields

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E}. \tag{7.16}$$

Alternative 7.1. The acceleration of the quasiparticle is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{v}}{v}\right) + \frac{q}{m}\mathbf{E}, \tag{7.17}$$

or, written out by components,

$$\begin{aligned} a_x &= \frac{\alpha}{v}\frac{\mathrm{d}v_x}{\mathrm{d}t} - \frac{\alpha v_x}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{qE}{m} \\ a_y &= \frac{\alpha}{v}\frac{\mathrm{d}v_y}{\mathrm{d}t} - \frac{\alpha v_y}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t}. \end{aligned}$$

Using the kinematic relation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v_x}{v}a_x + \frac{v_y}{v}a_y, \tag{7.18}$$

we may rewrite (7.17) as

$$\begin{cases} \left(1 - \frac{\alpha}{v} + \frac{\alpha v_x^2}{v^3}\right)a_x + \frac{\alpha v_x v_y}{v^3}a_y = \frac{qE}{m} \\ \left(1 - \frac{\alpha}{v} + \frac{\alpha v_y^2}{v^3}\right)a_y + \frac{\alpha v_x v_y}{v^3}a_x = 0, \end{cases} \tag{7.19}$$

which, in turn, may be rewritten without v_x and v_y as

$$\begin{cases} \left(1 - \frac{\alpha\sin^2\theta}{v}\right)a_x + \frac{\alpha\sin\theta\cos\theta}{v}a_y = \frac{qE}{m} \\ \left(1 - \frac{\alpha\cos^2\theta}{v}\right)a_y + \frac{\alpha\sin\theta\cos\theta}{v}a_x = 0. \end{cases}$$

This system of equations may be solved to obtain our desired result,

$$\begin{cases} a_x = \frac{qE}{m} + \frac{qE}{m}\frac{\alpha\sin^2\theta}{v-\alpha} \\ a_y = -\frac{qE}{m}\frac{\alpha\cos\theta\sin\theta}{v-\alpha}. \end{cases} \tag{7.20}$$

Alternative 7.2. In Cartesian coordinates,

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t}\cos\theta - p\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = qE \\ \dfrac{\mathrm{d}p}{\mathrm{d}t}\sin\theta + p\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 0, \end{cases} \tag{7.17}$$

from which we solve for the time derivatives and obtain

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = qE\cos\theta \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{qE}{p}\sin\theta. \end{cases} \tag{7.18}$$

Splitting (7.5) into components and differentiating with respect to time yields

$$\begin{cases} a_x = \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\left(\dfrac{p}{m}+\alpha\right)\cos\theta\right) = \dfrac{1}{m}\cos\theta\dfrac{\mathrm{d}p}{\mathrm{d}t} - \left(\dfrac{p}{m}+\alpha\right)\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \\ a_y = \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\left(\dfrac{p}{m}+\alpha\right)\sin\theta\right) = \dfrac{1}{m}\sin\theta\dfrac{\mathrm{d}p}{\mathrm{d}t} + \left(\dfrac{p}{m}+\alpha\right)\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}. \end{cases} \tag{7.19}$$

Substituting (7.18) into (7.19) and using (7.5) yields our final result,

$$\begin{cases} a_x = \dfrac{qE}{m} + \dfrac{qE}{m}\dfrac{\alpha\sin^2\theta}{v-\alpha} \\ a_y = -\dfrac{qE}{m}\dfrac{\alpha\cos\theta\sin\theta}{v-\alpha}. \end{cases} \tag{7.20}$$

Alternative 7.3. (7.15) can be decomposed into its components and written as

$$\begin{cases} p_x = m(v-\alpha)\cos\theta \\ p_y = m(v-\alpha)\sin\theta. \end{cases} \tag{7.17}$$

From (7.16) and (7.17) we obtain

$$\begin{cases} -m(v-\alpha)\sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} + m\cos\theta\dfrac{\mathrm{d}v}{\mathrm{d}t} = qE \\ m(v-\alpha)\cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} + m\sin\theta\dfrac{\mathrm{d}v}{\mathrm{d}t} = 0, \end{cases} \tag{7.18}$$

from which we solve for the time derivatives and obtain

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{qE}{m}\cos\theta \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{qE}{m}\dfrac{\sin\theta}{v-\alpha}. \end{cases} \tag{7.19}$$

Substituting (7.19) into the system

$$\begin{cases} a_x = \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}t}\cos\theta - \sin\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \\ a_y = \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{\mathrm{d}v}{\mathrm{d}t}\sin\theta + \cos\theta\dfrac{\mathrm{d}\theta}{\mathrm{d}t}, \end{cases}$$

we obtain our desired result,

$$\begin{cases} a_x = \dfrac{qE}{m} + \dfrac{qE}{m}\dfrac{\alpha\sin^2\theta}{v-\alpha} \\ a_y = -\dfrac{qE}{m}\dfrac{\alpha\cos\theta\sin\theta}{v-\alpha}. \end{cases} \tag{7.20}$$

Problem 8 (35 points)**.** When thermal radiation is incident upon a reflector, the reflector can do work on its surroundings with the radiation pressure supplied by the thermal radiation. This process can be studied by either using the principles of mechanics or those of thermodynamics. For simplicity, we model the thermal radiation as a one-dimensional beam of black-body radiation which is normally incident upon an ideal plane reflector. The radiation pressure acting upon the reflector is in balance with a resistive force such that the reflector undergoes uniform motion with speed v, in the same direction as the radiation. We are given the vacuum speed of light c and the spectral radiance[2] of one-dimensional black-body radiation as a function of the *frequency* ν and the black-body temperature T

$$\varphi(\nu, T) = \frac{2h\nu}{e^{h\nu/kT} - 1},$$

as measured in the laboratory reference frame, where h is the Planck constant and k is the Boltzmann constant.

(1) (14 points). We may conduct an analysis using mechanics. By considering the collision of the photons in the thermal radiation with the reflector, find the efficiency η of the reflector in converting the energy of the photons to the work done against drag, as observed in the lab frame.

(2) (15 points). An analysis with classical thermodynamics offers a different perspective. We may treat the setup as an ideal heat engine, with the reflector as the working substance. The incoming radiation can be modelled as the reflector absorbing heat from a hot reservoir, whereas the outgoing radiation can be modelled as the reflector releasing heat to a cold reservoir, with the reflector returning to its initial state. Using this model, show that the spectral radiance profiles of both the incoming and outgoing radiation, as observed in the reference frame of the reflector, fit that of black-body radiation, and find the efficiency of the reflector in the reflector frame.

(3) (6 points). Find the efficiency of the suggested heat engine as observed in the laboratory frame.

Solution 8:

(1) We consider the system composed of the reflector in uniform motion, as well as all the photons which collide with reflector *per unit time*. Let ε_1 and ε_2 (having the dimensions of power) be the total energy of these photons before reflection and after reflection respectively, and let F be the magnitude of the resistive force. The mechanical energy and momentum of the reflector remains the same before and after the collisions, and the ratio of the photons' energy to their momentum is the vacuum speed of light c, independent of their frequency. Thus, the impulse-momentum theorem and the work-energy theorem yields

$$-\frac{\varepsilon_1}{c} - \frac{\varepsilon_2}{c} = -F, \tag{8.1}$$

$$\varepsilon_2 - \varepsilon_1 = -Fv. \tag{8.2}$$

The required efficiency of conversion is given by

$$\eta = \frac{Fv}{\varepsilon_1}, \tag{8.3}$$

[2]Spectral radiance in frequency is the radiant flux received by a surface (in this case, the reflector) per unit frequency per unit time. For a more detailed explanation, see the Wikipedia article Irradiance.

which, when the value of ε_1 is obtained from solving (8.1) and (8.2) simultaneously and substituting into (8.3), yields

$$\eta = \frac{2v}{c+v}. \tag{8.4}$$

(2) Due to the relativistic Doppler effect (in-syllabus), the incident frequency ν_1 in the lab frame S is redshifted to ν_1' in the reflector rest frame S', and the relationship between the two frequencies is given by

$$\frac{\nu_1'}{\nu_1} = \sqrt{\frac{c-v}{c+v}}, \tag{8.5}$$

in which we note that the ratio of frequencies is independent of the incident frequency ν_1 and only dependent on the speed v of the reflector.

Suppose that during time interval $\mathrm{d}t_1$, for a cross-section of the path of incidence observed in frame S, a pulse of EM waves completely passes through this cross-section. When the pulse reaches the reflector, in frame S' the pulse takes time interval $\mathrm{d}t_1'$ for the whole length to be reflected. Since the number of periods in the pulse detected at the cross-section must be equal to that in the pulse detected at the reflector, we have

$$\nu_1 \,\mathrm{d}t_1 = \nu_1' \,\mathrm{d}t_1' \,. \tag{8.6}$$

Let $\varphi_1(\nu_1)$ and $\varphi_1'(\nu_1')$ be the spectral radiance profiles of the incident radiation as measured in frames S and S' respectively. Consider the number of photons passing through said cross-section in time interval $\mathrm{d}t_1$ and of frequency in the interval $[\nu_1, \nu + \mathrm{d}nu_1]$, *as observed in* S. All these photons arrive at the reflector and collide with it within time interval $\mathrm{d}t_1'$, so setting the numbers of photons in each case equal, we have

$$\frac{\varphi_1(\nu_1)\,\mathrm{d}\nu_1\,\mathrm{d}t_1}{h\nu_1} = \frac{\varphi_1'(\nu_1')\,\mathrm{d}\nu_1'\,\mathrm{d}t_1'}{h\nu_1'}. \tag{8.7}$$

Thus by (8.5), (8.6), and (8.7), we obtain

$$\frac{\varphi_1(\nu_1)}{\nu_1} = \frac{\varphi_1'(\nu_1')}{\nu_1'}. \tag{8.8}$$

In frame S', the incident frequency and reflected frequency are equal, i.e.

$$\frac{\nu_2'}{\nu_1'} = 1. \tag{8.9}$$

Noting that, in frame S', the numbers of periods in the incident and reflected EM waves are equal, the number of photons in each wave are equal, and that these hold true for all frequencies, we may argue in a manner similar to the derivation of (8.8) that the relationship between $\varphi_1'(\nu_1')$ and $\varphi_2'(\nu_2')$ is given by

$$\frac{\varphi_1'(\nu_1')}{\nu_1'} = \frac{\varphi_2'(\nu_2')}{\nu_2'}. \tag{8.10}$$

Let T_1 be the temperature corresponding to the incident black-body radiation. Then

$$\varphi_1 = \frac{2h\nu_1}{e^{h\nu_1/kT_1} - 1}. \tag{8.11}$$

By (8.5), (8.8), (8.9), (8.10), and (8.11), in frame S' the spectral radiance profiles of the incident and reflected radiation are given by

$$\varphi_1' = \frac{2h\nu_1'}{e^{h\nu_1'/kT_1'} - 1}, \tag{8.12}$$

$$\varphi_2' = \frac{2h\nu_2'}{e^{h\nu_2'/kT_2'} - 1}, \tag{8.13}$$

where T_1' and T_2' are given by

$$T_1' = \frac{\nu_1'}{\nu_1} T_1 = \sqrt{\frac{c-v}{c+v}} T_1, \tag{8.14}$$

$$T_2' = \frac{\nu_2'}{\nu_1'} T_1' = T_1' \tag{8.15}$$

respectively. Thus we have shown that these profiles are of the same form as that of black-body radiation of the appropriate temperature.

Since the heat engine is ideal, we may apply Carnot's theorem and obtain the extremely troll result of

$$\eta' = 1 - \frac{T_2'}{T_1'} = 0. \tag{8.16}$$

This wholly unexpected (and troll) result (to a contestant battered into submission by the previous problems and parts) is actually obvious in hindsight and in line with the predictions of mechanics, for in the rest frame of the reflector, the displacement of the reflector is zero, so it does no work on the surroundings; thus the work done using the energy of the photons must also be zero.

(3) The observed work done and photon energy are both dependent on the chosen reference frame, so the observed efficiency in the reflector frame and the lab frame must also necessarily be different. This discrepancy can be reconciled by noting that the reflected frequency ν_2' as observed in frame S' is redshifted to ν_2 in frame S, so we have

$$\frac{\nu_2}{\nu_2'} = \sqrt{\frac{c-v}{c+v}}. \tag{8.17}$$

By a similar argument as the derivation of (8.12), we obtain the spectral radiance profile of the reflected radiation as observed in S,

$$\varphi_2 = \frac{2h\nu_2}{e^{h\nu_2/kT_2} - 1}, \tag{8.18}$$

which has the same form as black-body radiation of temperature

$$T_2 = \frac{\nu_2}{\nu_2'} T_2' = \frac{c-v}{c+v} T_1, \tag{8.19}$$

whose derivation involves (8.14), (8.15), and (8.17). Finally, by Carnot's theorem, we obtain the efficiency

$$\eta = 1 - \frac{T_2}{T_1} = \frac{2v}{c+v}, \tag{8.20}$$

which agrees with the result derived using mechanics.