Chinese Physics Olympiad 2018 Finals Theoretical Exam

Translated By: **Wai Ching Choi**
Edited By: **Kushal Thaman**

Problem 1 (35 points)**.** Refer to Figure 1.1. A solid hemisphere of radius R and mass M lies at rest upon a smooth tabletop. A smaller solid sphere of uniform density, mass m, and radius r rests upon the apex of the hemisphere. At some instant, the sphere is given a small perturbation and begins to move along the surface of the hemisphere. In the course of the sphere's motion, its position with respect to the hemisphere is described by the angle θ, where θ is the angle between the vertical and the line joining the centres of each body. We are given the moment of inertia of the sphere $\frac{2}{5}mr^2$ about its axis of symmetry, the coefficient of *kinetic* friction μ between the sphere and hemisphere, the assumption that the maximum static friction is equal to the kinetic friction, and the gravitational acceleration g.

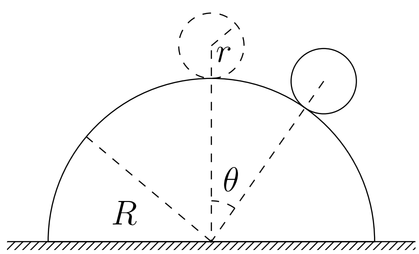

Figure 1.1: A sphere rolling down a hemisphere.

(1) (15 points). The sphere rolls without slipping for a while after it begins to move. When $\theta = \theta_1$, find the magnitudes of the *hemisphere's* velocity $V_M(\theta_1)$ and its acceleration $a_M(\theta_1)$ during this motion.

(2) (15 points). The sphere begins to slip when $\theta = \theta_2$. Find a condition, involving $V_M(\theta_2)$ and $a_M(\theta_2)$, satisfied by θ_2.

(3) (5 points). The sphere loses contact with the hemisphere when $\theta = \theta_3$. Find the speed of the centre of mass of the sphere relative to the hemisphere.

Problem 2 (35 points)**.** Both plates (1 and 2) of a parallel-plate capacitor have area S, are fixed horizontally with separation d, and are connected to the circuit shown in Figure 2.1, where U is the emf generated by the power source. An uncharged conducting plate (3) of mass m and having the same dimensions as plates 1 and 2 is placed atop plate 2 and contacts it well. The whole setup is placed inside a vacuum chamber with vacuum permittivity ε_0. When the switch K is closed, plate 3 colliides with plates 1 and 2 in an alternating fashion and undergoes reciprocating motion. We make the following assumptions: the electric field between 1 and 2 is uniform; the resistance of the wires and the internal resistance of the cell are small, so the characteristic charging and discharging times can be neglected; when plate 3 makes contact with plates 1 or 2, the free charge within the contacting plates reaches equilibrium instantly; and all collisions are inelastic. The gravitational acceleration is g.

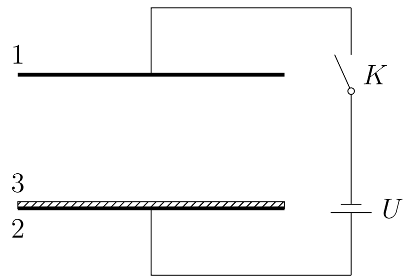

Figure 2.1: A bouncing metal plate.

(1) (17 points). Find the minimum possible value of U.

(2) (18 points). Find the period of the reciprocating motion plate 3 is undergoing.

Note. The integral

$$\int \frac{dx}{\sqrt{ax^2+bx}} = \frac{1}{\sqrt{a}} \ln\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx}\right)+C$$

is given, where $a>0$ and C is a constant of integration.

Problem 3 (35 points)**.** Refer to Figure 3.1. An inextensible, massive string of uniform linear density λ is threaded through a disc-shaped, fixed pulley of radius R, whose axle is a distance L from the floor. The system is initially at rest. When $t = 0$, the pulley acquires a constant angular speed ω (which is maintained throughout) in the anticlockwise direction and causes the string to move as well. The coefficient of kinetic friction between the pulley and the string is μ. The suspended parts of the string are vertical throughout the motion, the ends of the string never leave the floor, and the piles of string resting on the floor are concentrated at two points. We are given the gravitational acceleration g. Denote the tension at the points on the left and right hand sides of the pulley, where the pulley is tangent to the string, by T_1 and T_2 respectively.

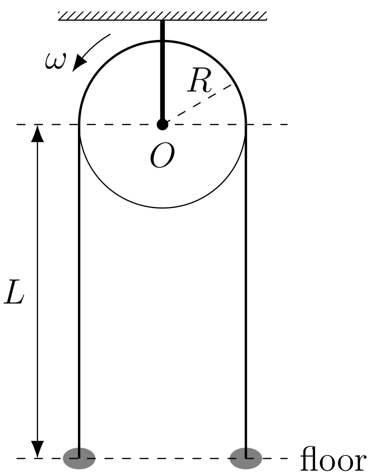

Figure 3.1: A massive string threaded through a rotating pulley. The grey blobs indicate piles of string.

(1) (20 points). Obtain a system of dynamical equations for any short length of string at all possible locations on the string. Consider two cases: the suspended parts and the part threaded through the pulley.

(2) (15 points). Find the maximum possible speed of the string.

Note. The identities

$$\frac{dy}{dx} + \alpha y = e^{-\alpha x}\frac{d(ye^{\alpha x})}{dx},$$

$$\int e^{\alpha x}\cos x\, dx = \frac{e^{\alpha x}}{1+\alpha^2}(\alpha\cos x + \sin x) + C_1, \quad \text{and}$$

$$\int e^{\alpha x}\sin x\, dx = \frac{e^{\alpha x}}{1+\alpha^2}(\alpha\sin x + \cos x) + C_2$$

are given, where C_1 and C_2 are constants of integration.

Problem 4 (35 points)**.** Refer to Figure 4.1. A taut string of length L is placed along the x-axis, whose left end is located at the origin. Both ends of the string can be attached to a vibration generator, which drives oscillations in the y-direction. The speed of wave propagation is u.

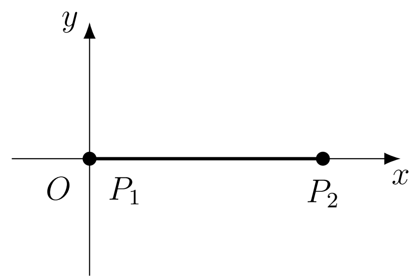

Figure 4.1: A vibrating string.

(1) (22 points). We fix the right end of the string P_2 and connect the left end P_1 to the generator. When the system reaches a steady state, the displacement of the left end is given by $y(x=0,t) = A_0 \cos(\omega t)$, where A_0 and ω are the amplitude and angular frequency of the oscillation respectively.

(i) (10 points). We are given that the transverse oscillation attenuates down the string with coefficient $\gamma > 0$. Find the oscillation amplitude everywhere on the string, given that, for a string of infinite length, the equation of a transverse wave travelling and attenuating in the positive x direction is given by $y(x,t) = Ae^{-\gamma x} \cos(\omega t - \omega x/u + \varphi)$, where A and φ are the amplitude and initial phase of the oscillation at $x = 0$ respectively.

(ii) (12 points). We now ignore the effects of attenuation. Find the equation of the standing wave on the string. Find also the positions of the nodes and antinodes of the standing wave.

(2) (13 points). We connect both ends of the string to the generator, such that the displacements of P_1 and P_2 are given by $y(x=0,t) = A_0 \cos \omega t$ and $y(x=L,t) = A_0 \cos(\omega t + \varphi_0)$ respectively. Ignoring the effects of attenuation, find the equation of the wave everywhere on the string for the cases $\varphi_0 = 0$ and $\varphi_0 = \pi$ respectively, and state the condition for the resonance frequency ω in each case.

Problem 5 (35 points)**.** An insulated, thin-walled container of mass M is placed in outer space, far away from any celestial bodies, such that outer space can be modelled as a vacuum. The initial velocity of the container is zero when observed in a certain inertial reference frame. The capacity of the container is V, and it is initially filled with N_0 molecules of a monatomic ideal gas, whose individual mass is m, and whose initial temperature is T_0. When $t = 0$, the container is punctured, and a hole of area S appears on the wall. The container begins to move but does not rotate due to the gas leak. We assume that the hole is small and that the ideal gas remains in thermodynamic equilibrium throughout the process. We are given the Maxwell-Boltzmann distribution function

$$f(v_x) = \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv_x^2}{2kT}\right)$$

for the x-component of the molecular velocity v_x, where k is the Boltzmann constant. Find

(1) the number of molecules escaping the container per unit time, in terms of the molecular number density n and the temperature T of the gas at that instant;

(2) the average kinetic energy of each molecule relative to the container, in terms of T;

(3) the temperature of the gas at time t; and

(4) the speed of the container at time t.

Note. We are given the identities

$$\int_0^\infty xe^{-Ax^2}\,dx = \frac{1}{2A},$$
$$\int_0^\infty x^2e^{-Ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{A^3}}, \quad \text{and}$$
$$\int_0^\infty x^3e^{-Ax^2}\,dx = \frac{1}{2A^2}.$$

Problem 6 (35 marks)**.** The refractive index n of an optical medium can either be greater than or less than zero. Media in which $n < 0$ are called negative-index meta-materials (NIMs). When light propagates in a NIM, its optical path length is negative.[1] If we say that the angle of refraction is negative when the incident ray and refracted ray are on the same side of the normal, Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

would still hold, even if the refractive index on either side were negative. Here, n_1 and n_2 can either be positive or negative, and θ_2 is the angle of refraction. We use the convention where $\theta_1 \geq 0$ always.

(1) (10 marks). Suppose that a beam of light is incident upon an interface between two materials with different refractive indices. For each possible case as depicted in Figures 6.1 and 6.2, show the path of the rays entering material 2 and the corresponding wavelets on the figures. Hence, show that the generalised Snell's law holds.

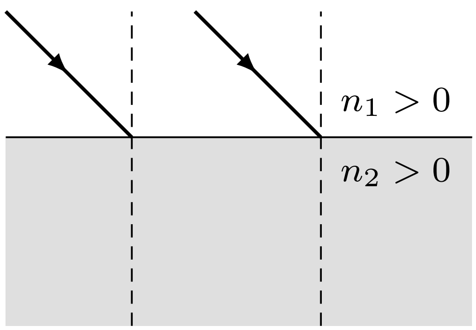

Figure 6.1

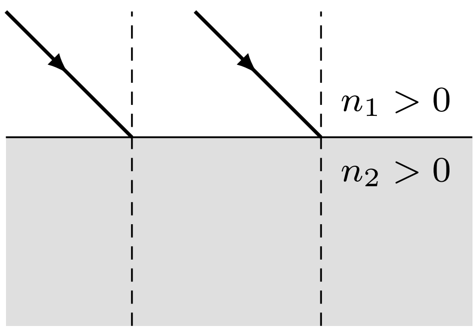

Figure 6.2

(2) (13 points). Refer to Figure 6.3. A spherical surface of radius R and centre C partitions three-dimensional space into an outer region and an inner region, with refractive indices $n_1 > 0$ and $n_2 < 0$ respectively. Considering any optical axis passing through C, we set the origin at O, the intersection of the axis with the interface. Light ray x is incident upon the interface at M and refracts to give ray y, as shown. Let s_1 and s_2 be the object distance and image distance respectively. Under the paraxial approximation, derive the equivalent lens equation (a relationship between s_1, s_2, and the given parameters of the system) and an expression for the horizontal magnification of an image. Note down the sign of each quantity in the final results *explicitly*.

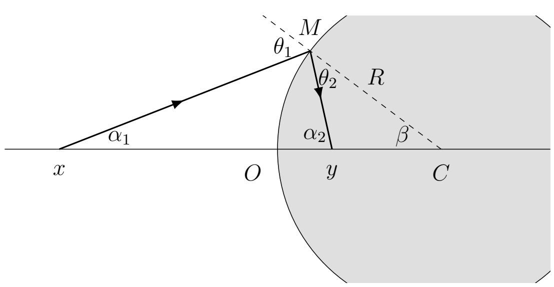

Figure 6.3: A spherical NIM.

[1] When we say that the optical path length of a light ray is negative, we mean that, as the light propagates, its phase change is the opposite of the case where it is propagating in a typical medium. In other words, we can model a light ray travelling in a NIM of some refractive index $n_2 < 0$ by treating it as if it were travelling in the *opposite* direction in a medium of refractive index $-n_2 > 0$.

(3) (12 points). Suppose that medium 1 is air, i.e. $n_1 \approx 1$, and n_2 can either take positive or negative values. We place a thin convex lens of focal length $f = 1.5R$ in front of the spherical interface such that its optical axis passes through C, one focal point is inside the NIM, and the distance between O and the centre of the lens O' is d. A beam of light rays all parallel to the axis is incident upon the lens. For each set of parameters in Table 6.1, obtain the distance between O and the point at which all light rays converge. Also draw a figure illustrating the path of the light rays in case 4.

Case	n_2	d
1	1.5	$0.35R$
2	1.5	$0.85R$
3	-1.5	$0.35R$
4	-1.5	$0.85R$

Table 6.1

Problem 7 (35 points)**.** Modelling the physical behaviour of solids, which often have complicated internal structures, can be quite challenging using conventional means alone. If we wish to simplify the problem while still taking the interactions between constituent particles into account, we may use the concept of *quasiparticles*, for which the energy-momentum relation may be different from the one which usually applies to real particles. When external electric or magnetic fields are applied on a solid, the motion of quasiparticles can typically be treated with the methods of classical mechanics.

One type of quasiparticle of effective mass m and carrying charge q exists within some two-dimensional interface-like structure. Its motion is constrained to the xy-plane. Its kinetic energy K can be expressed in terms of the magnitude of its momentum p by the equation

$$K = \frac{p^2}{2m} + \alpha p$$

where α is a positive constant.

(1) (4 points). For a *real* particle of mass m in free motion, its kinetic energy K can be expressed in terms of the magnitude of its momentum p by $K = p^2/2m$. Express its velocity $\mathbf{v}$ in terms of its momentum $\mathbf{p}$ using the work-energy theorem.

(2) (5 points). Using a similar method, express the velocity $\mathbf{v}$ of a *quasiparticle* in terms of its momentum $\mathbf{p}$.

(3) (4 points). Express $v = |\mathbf{v}|$ in terms of K.

(4) (11 points). We now place the two-dimensional interface within a uniform magnetic field of magnitude B and pointing in the $+z$ direction. For a quasiparticle of kinetic energy K, which will undergo uniform circular motion, find the radius of its trajectory, the period of its motion, and the magnitude of its angular momentum.

(5) (11 marks). We replace the magnetic field with a uniform electric field of magnitude E and pointing in the $+x$ direction. Note that the component of the quasiparticle's acceleration perpendicular to the electric field may be nonzero. Find the components of the quasiparticle's acceleration a_x and a_y when it moves with speed v and its velocity makes an angle θ with the electric field.

Problem 8 (35 points)**.** When thermal radiation is incident upon a reflector, the reflector can do work on its surroundings with the radiation pressure supplied by the thermal radiation. This process can be studied by either using the principles of mechanics or those of thermodynamics. For simplicity, we model the thermal radiation as a one-dimensional beam of black-body radiation which is normally incident upon an ideal plane reflector. The radiation pressure acting upon the reflector is in balance with a resistive force such that the reflector undergoes uniform motion with speed v, in the same direction as the radiation. We are given the vacuum speed of light c and the spectral radiance[2] of one-dimensional black-body radiation as a function of the *frequency* ν and the black-body temperature T

$$\varphi(\nu, T) = \frac{2h\nu}{e^{h\nu/kT} - 1},$$

as measured in the laboratory reference frame, where h is the Planck constant and k is the Boltzmann constant.

(1) (14 points). We may conduct an analysis using mechanics. By considering the collision of the photons in the thermal radiation with the reflector, find the efficiency η of the reflector in converting the energy of the photons to the work done against drag, as observed in the lab frame.

(2) (15 points). An analysis with classical thermodynamics offers a different perspective. We may treat the setup as an ideal heat engine, with the reflector as the working substance. The incoming radiation can be modelled as the reflector absorbing heat from a hot reservoir, whereas the outgoing radiation can be modelled as the reflector releasing heat to a cold reservoir, with the reflector returning to its initial state. Using this model, show that the spectral radiance profiles of both the incoming and outgoing radiation, as observed in the reference frame of the reflector, fit that of black-body radiation, and find the efficiency of the reflector in the reflector frame.

(3) (6 marks). Find the efficiency of the suggested heat engine as observed in the laboratory frame.

[2]Spectral radiance in frequency is the radiant flux received by a surface (in this case, the reflector) per unit frequency per unit time. For a more detailed explanation, see the Wikipedia article Irradiance.