Chinese Physics Olympiad 2019 Finals Theoretical Exam

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Problem 1 (40 points)**.** A new car model must pass destructive testing before it can be approved for sale. In one particular test, a three-spoked wheel has one of its spokes knocked out. Its vertical cross section is shown in Figure 1.1. The wheel can be modelled as a combination of an annular lamina and two identical rods, all of which have uniform density. The lamina has an outer radius of R_0 and an inner radius of $R_1 = \frac{4}{5}R_0$. The mass of each rod is m and the mass of the lamina is $M = 8m$. The wheel is released from rest from the position shown in the figure, after which it rolls without slipping. Find

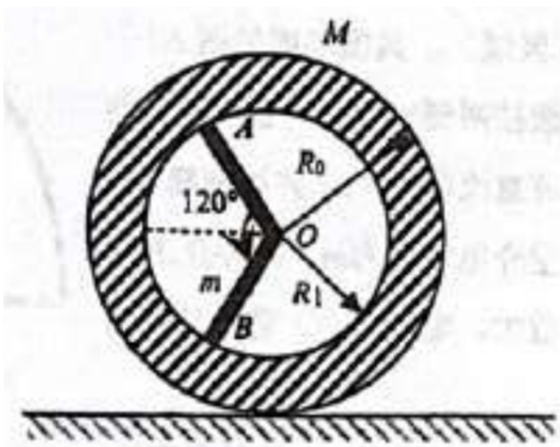

Figure 1.1: A broken wheel.

(1) the initial angular acceleration α_0 of the wheel; and

(2) the angular acceleration α of the wheel, the normal force F_N, and friction F_f acting on the wheel by the ground, when the spoke OB first reaches a vertical position.

Problem 2 (40 points)**.** Consider a cylinder-like solid whose base is described by a half-annulus of inner radius a and outer radius b. The solid is composed of two leaky dielectrics[1], with parameters ϵ_{r1} and σ_1 for the region $0 \leq \varphi < \theta_0$ and parameters ϵ_{r2} and σ_2 for the region $\theta_0 < \varphi \leq \pi$, as shown in Figure 2.1. Now we coat each flat rectangular face of the solid with a metallic film, apply a steady potential V_0 between the two conductors (see figure for polarity), and wait until the system reaches a steady state. We are given the vaccum permittivity ϵ_0, that ϵ_{r1} and ϵ_{r2} are large, and that σ_1 and σ_2 are small. Neglect any fringe effects. Take $V = 0$ on the left metallic film.

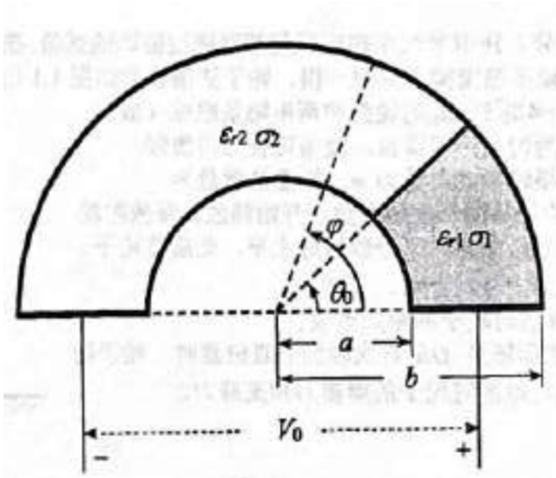

Figure 2.1: A leaky dielectric.

(1) Find the magnitude of the electric field E and the electric potential V everywhere within the dielectric.

(2) Find the total accumulated charge Q at $\varphi = \theta_0$.

(3) Obtain the resistance and capacitance across the regions $0 \leq \varphi < \theta_0$ (R_1, C_1) and $\theta_0 < \varphi \leq \pi$ (R_2, C_2).

(4) Suppose we disconnect the potential source at time $t = 0$. In order to model the subsequent behaviour, we can design a circuit based on the given parameters of the solid. Draw this circuit. Hence, find the time dependence $V(t)$ of the potential difference between the metal films on each end.

[1] A leaky dielectric is a dielectric which is not a perfect insulator. Such a material can be described by two parameters: its relative permittivity ϵ_r and its conductivity σ.

Problem 3 (40 points). We investigate the applications of the superconducting gravimeter in this problem. The superconducting gravimeter can be used to obtain information about the Earth's interior such as mantle dynamics, distribution of mass, crustal tides, groundwater, and distribution of minerals. It can also be used to monitor external phenomena such as the coastal tides and climate change. These data can be obtained through long-term, high-precision measurements of minute changes in the local gravitational acceleration using the superconducting gravimeter. The superconducting gravimeter is extremely sensitive (with an uncertainty of around $10^{-9}g$) and has good measurement stability, and is thus the primary instrument of Earth scientists wishing to measure the local gravitational acceleration accurately. New opportunities arise for the the development of Earth sciences and precision measurements of the gravitational force, as more superconducting gravimeters and communication networks are built around the world. The following questions concern the effects of ground water on the local gravitational acceleration and the operating principles of the superconducting gravimeter.

1. We model the Earth as a uniform solid sphere of radius 6370 km, whose gravitational acceleration is $g_0 = 9.80\,\mathrm{m\,s^{-2}}$. Suppose that a spherical underground lake of diameter 20 km whose centre is 15 km from the surface. Find the difference in gravitational acceleration $\Delta g = g' - g_0$ on the surface, vertically above the centre of the lake, given the values of the gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ and the density of water $\rho_\mathrm{w} = 1.0 \times 10^3\,\mathrm{kg\,m^{-3}}$. Neglect the effects due to Earth's rotation.

2. The main component of the superconducting gravimeter is a spherical shell made of a cooled, superconducting niobium alloy, suspended in a magnetic field. *For convenience, we model the shell as a superconducting ring* (see Figure 3.1). The external magnetic field is supplied by a superconducting Helmholtz coil-like device, composed of an upper coil with turn number αN and a lower coil with turn number N, where $\alpha < 1$. The coils share the same radius R and axis of symmetry, are separated by R, and are connected in series to an external power source. Before a measurement begins, the currents in the Helmholtz coil and the ring are zero, the ring is in the same plane as the lower coil ($z = 0$), and it shares the same axis of symmetry with both coils. When a measurement is made, the power source slowly introduces a current i_0 to the Helmholtz coil, which induces a current in the ring. If we adjust the magnitude of i_0, we can make the ring levitate such that it rests in the same plane as the *upper* coil ($z = R$), as shown in Figure 3.1. As both the coil and the ring are superconductors (at a temperature of 4.2 K), steady currents can be maintained in both conductors, even if the power source is switched off.

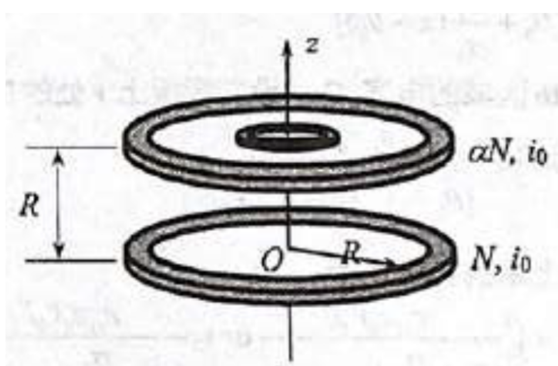

Figure 3.1: A model of the superconducting gravimeter.

We are given the mass of the ring m, its diameter $D = 2R$ (where $D \ll 2R$), and its self-inductance L; and that the radial and vertical components of the magnetic field generated by the Helmholtz coil can be expressed as

$$\begin{cases} B_z = B_0[1 - \beta(z - R)], \\ B_r = \dfrac{1}{2}\beta B_0 r, \end{cases}$$

where B_0 is the magnetic flux density at the centre of the upper coil, β is the radial coefficient, and r is the distance to the z-axis. Express

(1) B_0 and β;

(2) the induced current I_0 in the ring when it is in equilibrium at $z = R$; and

(3) the local gravitational acceleration g

in terms of the given parameters of the coil and the ring, and the current i_0 in the coil.

3. We consider the gravimeter after the power source has been removed. In order to obtain precise measurements of the minute changes in the local gravitational acceleration, an ac bridge circuit is used in the construction of the superconducting gravimeter. To understand its operating principle and simplify the calculations involved, we make a number of approximations. We model the ring as a thin conducting plate of area A, and fix two aluminium plates, each also of area A, above and below the ring in a symmetrical manner (i.e. the plates are located at positions $z = R + \frac{d}{2}$ and $z = R - \frac{d}{2}$). The three plates combined can be modelled as two capacitors, one (C_1) composed of the upper plate and the superconductor, and another (C_2) composed of the lower plate and the superconductor, as shown in Figure 3.2. We then connect C_1 and C_2 to two inductors L_0 and capacitors C_0 to form a bridge circuit, and then connect it to an a.c. power source (whose frequency is far greater than the rate of change of the local gravitational acceleration), as shown in Figure 3.3. Suppose that at some instant, the local gravitational acceleration is g, the ring is at $z = R$, and the bridge circuit is balanced. At this instant, we apply an additional steady voltage V_C across each capacitor C_1 and C_2. When the gravitational acceleration changes by Δg, the bridge circuit becomes unbalanced. If we increase the voltage across C_1 by ΔV and decrease the voltage across C_2 by ΔV, the bridge circuit becomes balanced again.

Find the relationship between Δg and ΔV. Neglect all fringe effects.

Figure 3.2: The simplified model. Translation (from top to bottom): upper plate, ring, lower plate.

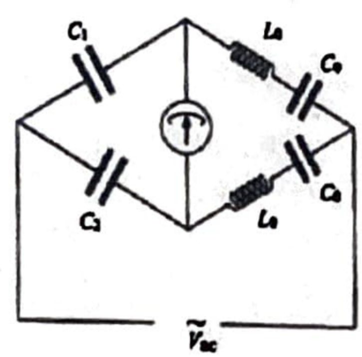

Figure 3.3: The external circuit. C_1 and C_2 on the left, L_0 and C_0 on the right, $\tilde{V}_{\rm ac}$ at the bottom.

Problem 4 (40 points)**.** An automobile is powered by the combustion of fuel in air occurring within combustion chambers in its engine. The power output of the engine is proportional to the *mass* of air taken in by the combustion chambers. Many automobiles are fitted with turbochargers (which compress air and thus increase air density) in order to maximise their power output. Automobiles may also be fitted with intercoolers, which cool the compressed air before it enters the combustion chamber, to further increase the air density. In a typical device composed of a turbocharger and an intercooler, air enters the turbocharger at pressure $p_0 = 1.01 \times 10^5\,\mathrm{Pa}$ and temperature $T_0 = 15\,°\mathrm{C}$. The turbocharger makes the air undergo adiabatic compression and reach a pressure of $p_1 = 1.45 \times 10^5\,\mathrm{Pa}$. The air then passes through the intercooler, which cools the air at constant pressure to the external temperature T_0, after which the air fills the combustion chamber. Model the air as an ideal gas. We are given the volume $V_0 = 400\,\mathrm{cm}^3$ of the chamber, the molar mass $M = 29.0\,\mathrm{g\,mol^{-1}}$ of the air, the ratio of specific heats $\gamma = 7/5$, and the universal gas constant $R = 8.31\,\mathrm{J\,K^{-1}\,mol^{-1}}$.

(1) Find the mass of air m_1 injected into the combustion chamber and compare it with the case where the device was not installed (m_0). Write down the ratio $R_1 = m_1/m_0$.

(2) Find the mass of air m_2 injected into the chamber for the case where no intercooler is present. Write down the ratio $R_2 = m_2/m_0$.

(3) Find the work done on the injected air by the turbocharger (W_1) and the intercooler (W_2). Find also the entropy change of the air due to the turbocharger (S_1) and the intercooler (S_2).

Problem 5 (40 points)**.** The 2018 Nobel Prize in Physics was awarded in part to Arthur Ashkin for his pioneering work on optical tweezers. Optical tweezers use focused laser beams to trap small particles (see Figure 5.1 for a schematic). The helium-neon (He-Ne) laser emits a beam with wavelength $\lambda = 632.8\,\text{nm}$, beam diameter $D_1 = 1.5\,\text{mm}$, and power $P = 30\,\text{mW}$. To enhance the trapping effect, we may place a beam expander in the path of the laser beam (which increases the beam diameter to D_2). The expanded beam reflects off of a mirror and is focused by a lens of focal length $f = 4\,\text{mm}$. It finally passes into a tank of distilled water, in which a certain quantity of spherical polystyrene particles are placed. We are given the following parameters: the diameter $D_b = 2.0\,\text{um}$ of one particle and its density $\rho_b = 1.0 \times 10^3\,\text{kg}\,\text{m}^{-3}$, the refractive index $n_g = 1.46$ of the glass tank wall, the refractive index $n_w = 1.33$ of water, the thickness $d = 1.0\,\text{mm}$ of the glass base, the distance $a = 2.0\,\text{mm}$ between the focusing lens and glass base, the ambient temperature $T = 300\,\text{K}$, the refractive index $n_a = 1.0$ of air, the gravitational acceleration $g = 9.80\,\text{m}\,\text{s}^{-2}$, and the Boltzmann constant $k_B = 1.38 \times 10^{-23}\,\text{J}\,\text{K}^{-1}$. When a particle happens to cross the path of the laser beam, it may be trapped by the beam.

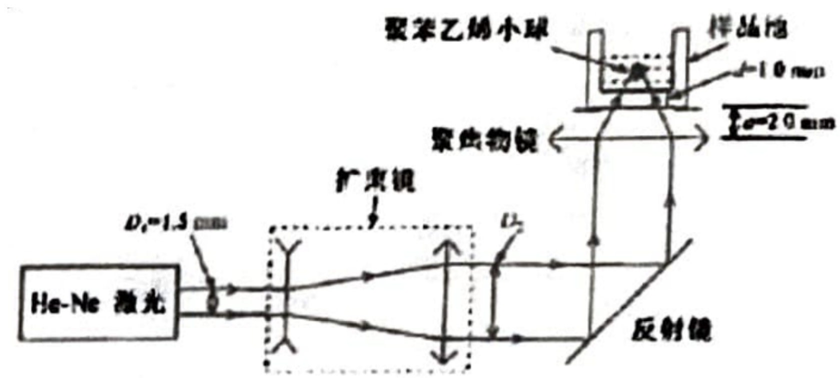

Figure 5.1: A typical optical tweezer setup. Translation (top to bottom, left to right): polystyrene particle, sample tank, $d = 1.0\,\text{mm}$, $a = 2.0\,\text{mm}$, focusing lens, beam expander, $D_1 = 1.5\,\text{mm}$, D_2, He-Ne laser, reflector.

(1) In the case where the beam expander is not used, find the diameter D_{o1} of the light spot after the beam passes through the focusing lens.

(2) We use the beam expander hereafter. As shown in Figure 5.1, the expander consists of a concave lens of focal length $f_1 = -6.3\,\text{mm}$ and a convex lens of focal length $f_2 = 25.2\,\text{mm}$. Find the diameter D_2 of the expanded beam, given that the constituent rays of the expanded beam are all parallel to each other.

(3) We now focus the beam to a point inside the sample tank. Find, using the paraxial approximation, the distance of the effective focal point from the upper surface of the glass base and the diameter of the beam at the focal point.

(4) Figure 5.2 shows a model of the effects of radiation pressure due to an optical gradient on a polystyrene particle immersed in water. The gradient is modelled with two light rays of different intensities. We treat the system using geometric optics.

Light rays L_1 (power $P_1 = 1.0\,\text{mW}$) and L_2 (power $P_2 = 2.0\,\text{mW}$) travelling parallel to the x-axis fall upon the particle. Both rays are refracted by the particle and make angles with the x-axis as shown. Neglecting all losses due to reflection and absorption, find the optical-gradient force $\mathbf{F}_b$ on the particle. Also write down the values of the ratios $\eta_x = F_{bx}/mg$ and $\eta_y = F_{by}/mg$.

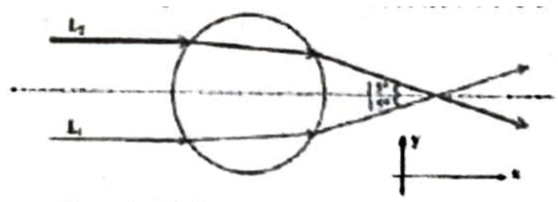

Figure 5.2: Modelling radiation pressure with geometric optics. L_1 is the weak lower ray, while L_2 is the strong upper ray. Each refracted ray makes an angle of 15° with the x-axis.

(5) Find the rms speed v_p of each polystyrene particle at room temperature 300 K.

(6) We can also model the behaviour of the optical tweezers with the potential well produced by the optical gradient. The radial potential distribution due to the focused laser beam is given by

$$U(r) = \begin{cases} -U_0\left(1 - \frac{r^2}{b^2}\right) & (r \leq b), \\ 0 & (r > b), \end{cases}$$

where b is a constant, r is the distance to the centre of the optical field, and $U_0 = 0.078\,\text{eV}$. Find the maximum possible speed v_m of a particle which can be trapped by the potential well.

(7) The probability density function of the particle speed is given by

$$f(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right),$$

where m and v are the mass and speed of each particle respectively and T is the absolute temperature. Suppose that two types of polystyrene particles are suspended in the sample tank, one of mass m and another of mass $2m$. Compare the probabilities of each type of particle being trapped by the laser, P_m and P_{2m} respectively.

Problem 6 (80 points)**.** The starry night has inflamed human imaginations for much of our history. As science and technology make progress over the years, human knowledge about the cosmos has expanded substantially, especially in light of the new insights of the 21$^{\text{st}}$ century, such as the resolution of the missing neutrino problem, direct observation of gravitational waves, and the first image of a black hole. These events have inspired interest among members of the public. How can humanity explain cosmic and stellar phenomena, when our means are limited by the size and nature of our apparatus? Can the public, whose knowledge is limited to the high school level, quantitatively understand both the internal and external features of stars, such as their size, their mass, their life cycle, their structure, and their mechanisms of power generation? This problem, in constructing a model of the Sun, progresses from simple to complex models and follows the arc of history. As we cannot directly observe the Sun's interior, the models we consider must be tested by a manifold of observations, such as the Sun's age, its radius, its surface temperature, its power output, and its emission of neutrinos. All these observations must conspire to produce an internally-consistent model.

The following parameters are given: the radius of the Earth is 6370 km, the gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$, the Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$, the Boltzmann constant $k_B = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$, the elementary charge $e = 1.602 \times 10^{-19}\,\mathrm{C}$, the electron mass $m_\mathrm{e} = 0.51\,\mathrm{MeV}/c^2$, the proton mass $m_\mathrm{p} = 938.3\,\mathrm{MeV}/c^2$, the deuteron mass $m_\mathrm{D} = 1875.6\,\mathrm{MeV}/c^2$, the ^{3}He mass $m_{^3\mathrm{He}} = 2808.4\,\mathrm{MeV}/c^2$, and the ^{4}He mass $m_{^4\mathrm{He}} = 3727.5\,\mathrm{MeV}/c^2$.

1. A 2000-year-old Chinese classic text on arithmetic, the *Zhoubi Suanjing*, describes a method of measuring the ratio between the solar diameter D and the distance between the Sun and Earth $S_{\text{S-E}} = 1\,\mathrm{AU}$. Take a long, straight bamboo rod and hollow out its interior such that the inner diameter is d. Line up the axis of the rod with the Sun and adjust its length such that the solar disc just 'fills up' the end of the rod opposite to the observer.[2]

   (1) Write down the relationship between the angular diameter of the solar disc $\Phi = D/S_{\odot\oplus}$, d, and L.

   (2) The rod should be long in order to minimise the experimental error. What other ways are there to further minimise the error?

   (3) We obtain the measurements $d = 1$ inch and $L = 8$ feet.[3] Find the value of Φ.

2. On rare occasions (such as on 6 June 2012), Earth, Venus, and the Sun nearly lie on the same line (i.e. are collinear). If we observe the Sun at these times, Venus appears as a small black spot moving slowly across the solar disc. This is known as a transit of Venus, and a schematic is shown in Figure 6.1. The British astronomer Edmond Halley once suggested in 1716 that the angular diameter and the distance between the Sun and the Earth could be measured by collecting data from observations of the transit of Venus around the world at the same time. We are given that Earth and Venus orbit the Sun in the same direction with periods $T_\oplus = 365.256$ days and $T_♀ = 224.701$ days respectively, that the inclination between the two orbital planes is small, and that Earth's rotation can be neglected.

   (a) Suppose that both Earth and Venus orbit the Sun in circular orbits. Find the ratio r_{VE} of the Sun-Venus distance $S_{\odot ♀}$ to the Sun-Earth distance $S_{\odot\oplus}$.

[2] *Warning.* We are not advocating that you try this yourself. Following the given instructions may well result in blindness.

[3] *Translator's note.* The Chinese paper specifies the measurements in ancient Chinese inches and feet for the sake of historical accuracy. It is given there that 10 inches make a foot (not 12 inches).

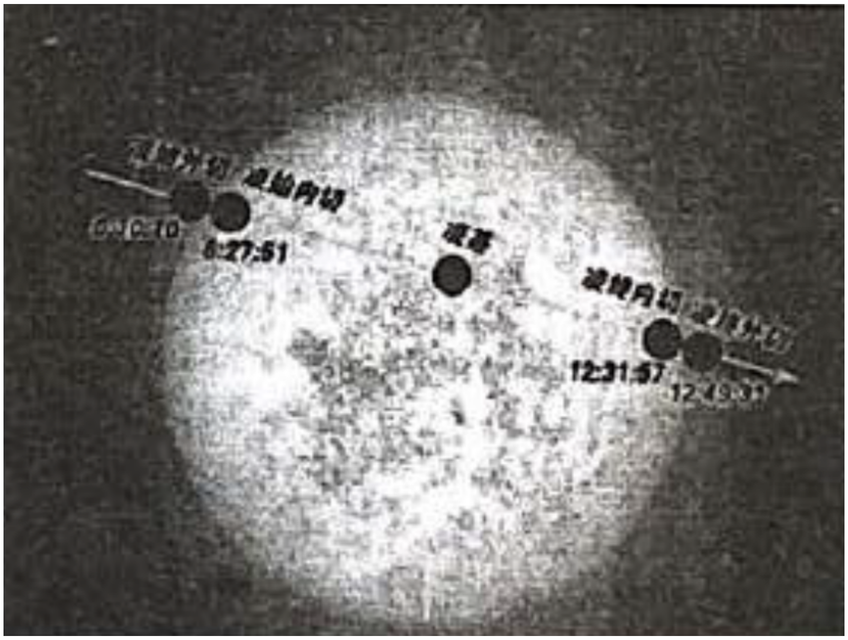

Figure 6.1: Transit of Venus on 6 June 2012. Translation (in order from left to right): Ingress, exterior, 6:10:10; Ingress, interior, 6:27:51; Greatest transit; Egress, interior, 12:31:57; Egress, exterior, 12:49:31.

(b) On 6 June 2012, the transit of Venus is observed somewhere on Earth, with the key phases of the transit marked on Figure 6.1. The shadow of Venus traces out a chord of the solar disc. Find the ratio between the length of this chord D_{P} and the Sun-Earth distance $S_{\odot\oplus}$. Find also the angular diameter Φ of the solar disc, given the distance $h_{\mathrm{P}} = 5D/16$ of the chord from its centre.

(c) The measurement of the distance between the Sun and Earth requires observing the same transit at different locations on Earth, as shown in Figure 6.2. Suppose that two observers P and P' located on the same line of longitude observe the transit simultaneously, separated by a distance H measured along the Earth's surface. P observes Venus tracing out a chord AB on the solar disc and measures a transit time of t_P, while P' sees the chord $A'B'$ and measures $t_{P'}$. Express $S_{\odot\oplus}$ in terms of Φ, r_{VE}, $T_{\oplus}$, $T_{♀}$, t_P, $t_{P'}$, and H.

(d) We are given the latitudes of Beijing, 39.5°, and Hong Kong, 22.5°, and their corresponding measured transit times, $t_P = $ 6:21:57 and $t_{P'} = $ 6:19:31. Use the given data and the result of (3) to obtain the numerical value of $S_{\odot\oplus}$.

3. The observations of the transit of Venus in 1882 yielded the results $S_{\odot\oplus} = 1.5 \times 10^8\,\mathrm{km}$ and $D = 1.4 \times 10^6\,\mathrm{km}$. The Earth receives solar radiation of intensity $I = 1.37\,\mathrm{kW\,m^{-2}}$, when measured in a plane perpendicular to the direction of radiation transfer. Obtain:

   (1) the total solar energy received by the Earth per unit time;

   (2) the ratio between the total energy consumption of all seven (7) billion people on Earth and the total solar energy, assuming that each person consumes the energy equivalent of three (3) metric tonnes of coal, and that 1 kg of coal produces 4 kWh of energy;[4]

   (3) the total energy emitted by the Sun per unit time; and

   (4) an estimate of the surface temperature of the Sun.

[4] *Translator's note.* This exercise was first done by Helmholtz in 1854, to demonstrate that the energy of the Sun could not be generated by chemical reactions.

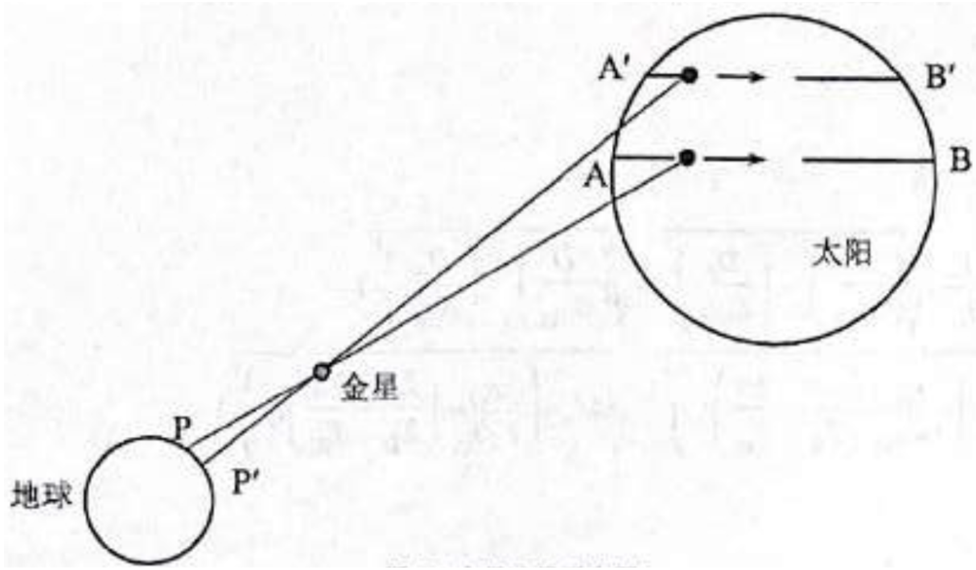

Figure 6.2: A method measuring the Sun-Earth distance. Translation (left to right): Earth, Venus, Sun.

4. (1) Obtain the solar mass and the average solar density.
   (2) If the energy emission of the Sun were caused by chemical processes, and assuming that the radiation emitted per unit mass were the same as the energy resulting from combustion of a unit mass of coal (see 3.2), for how long can the Sun continue to emit radiation in this manner?

   The Sun has existed as a star for five (5) billion years. How many times is the rate of energy production compared with the rate in the case where all the solar energy were accounted for by chemical reactions?

5. We know that the Sun is mainly composed of hydrogen due to spectral analysis. The fusion reactions hydrogen can be involved in are given by

$$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{D} + \mathrm{e}^+ + \nu_e; \tag{a}$$

$$\mathrm{D} + \mathrm{p} \longrightarrow {}^3\mathrm{He} + \gamma; \tag{b}$$

$${}^3\mathrm{He} + {}^3\mathrm{He} \longrightarrow {}^4\mathrm{He} + \mathrm{p} + \mathrm{p}. \tag{c}$$

   In the above reactions, the positrons will be annihilated, while the neutrinos will escape the Sun. The atomic nuclei are positively charged and repel each other, unless they are at a very high temperature, in which case their kinetic energy may be able to overcome the electrostatic repulsion between them, thereby undergoing a fusion reaction. When the temperature is high enough, all the atoms are ionised, become free nuclei and electrons, and form a plasma. In this section, we neglect the energy carried away by the neutrinos and the unreacted hydrogen carried away by the solar wind.

   (1) How many protons are consumed per unit time to maintain the solar radiation?
   (2) At present, 71% of the Sun is composed of hydrogen. Assuming that the rate just obtained is maintained ever after, for how long can this consumption be maintained? Write down the probability per unit time P_{pp} of a proton-proton reaction.
   (3) The number of neutrinos escaping the Sun does not fall off during transmission from the Sun to the Earth. How many neutrinos pass through a unit area on Earth, measuring in a plane perpendicular to the neutrinos' direction of travel?

(4) How much mass is lost by the Sun per unit time due to radiation?

6. The probability P of a fusion reaction is related to the plasma temperature T, and the probability also increases with the particle number density n. The relationship is given by $P = nR(T)$, where $R(T)$ is the rate of reaction. Figure 6.3 shows the relationship between the rate of reaction and the plasma temperature, for each of the three reactions given previously. The core is a spherical region whose boundary is concentric with the solar surface and whose radius is 1/4 the solar radius. Obtain an estimate for the lower limit of the solar core temperature.

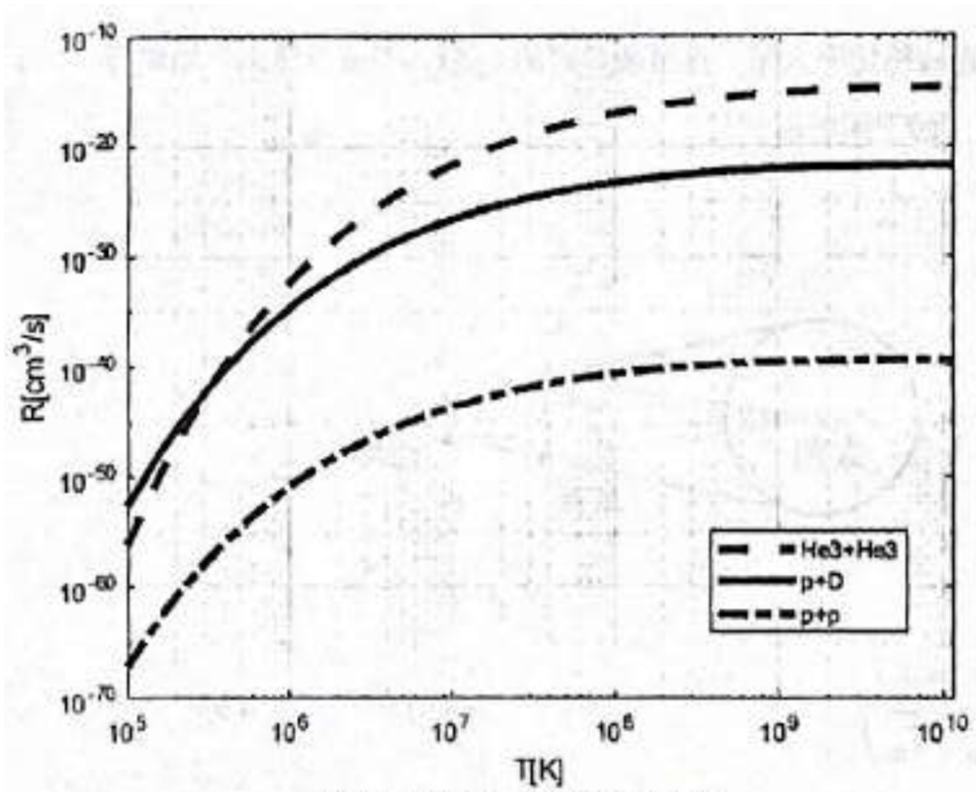

Figure 6.3: A graph of the rate of reaction R against the plasma temperature T.

7. Obtain an estimate of the maximum possible energy a single neutrino can carry away. Hence, obtain an estimate for the maximum relative error of the energy consumption estimate in 5.1, due to neglecting the neutrino energy loss.

8. We know from the previous tasks that the solar mass will decrease over time, which will affect the period and radius of Earth's orbit around the Sun. Obtain:

(1) the mass loss ΔM to radiation over a period of one (1) billion years; and

(2) the resulting change in Earth's orbital period.