Maxwell's Distribution for Physics Olympiads

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Abstract

We will talk about the probability theorem and learn about the Maxwell's distribution, and solve some problems on it.

Contents

1 Introduction

1.1 Information about Probability

Assume that we have a system having a really large number of particles. Also assume that our particles are characterized by some quantity, which can only have discrete values:

$$\upsilon_1, \upsilon_2, \ldots, \upsilon_n$$

Let us make a very large number of measurements (N) of the quantity υ, bringing the system before each measurement to the same initial state. Instead of performing repeated measurements of the same system, we can take N identical systems in the same state and measure the quantity υ once in all these systems. Such a set of identical systems in an identical state is called a *statistical ensemble*.

Let N_1 be the measurements that give result υ_1 and like so, N_i will be measurements for x_i. This is obvious that $\sum N_i = N$ which is total number of systems that ensemble. So the quantity $\dfrac{N_i}{N}$ called *relative frequency* which shows appearence of resuly υ_i, and if we take the so big amount of ensemble systems that ratio will give us **probability of appearance** of the result:

$$P_i = \lim_{N\to\infty} \frac{N_i}{N}$$

And from here if we take sum of both sides:

$$\sum P_i = \sum \lim_{N\to\infty} \frac{N_i}{N} = 1$$

since $\sum N_i = N$. With this definition we can also get the information about probability of being in two different quantity, namely:

$$P_{i \text{ or } k} = \frac{N_i + N_k}{N} = P_i + P_k$$

Furthermore assume that particles not only defined by υ_i also there is another quantity ε also characterizes the particles. Lets find the $P(\upsilon_i, \varepsilon_k)$ by the definition $N(\upsilon_i) = P(\upsilon_i)N$ also ε does not depend on υ so:

$$N(\upsilon_i, \varepsilon_k) = N(\upsilon_i)P(\varepsilon_k) = P(\upsilon_i)NP(\varepsilon_k)$$

So

$$P(\upsilon_i, \varepsilon_k) = P(\upsilon_i)P(\varepsilon_k) \tag{1}$$

We can also find the mean value of some quantity υ with knowing $P(\upsilon)$:

$$\langle \upsilon \rangle = \frac{\sum N_i \upsilon_i}{N} = \sum P(\upsilon_i)\upsilon_i$$

1.2 Probability distribution function

Let us go further and say that our quantity is not discrete. Here is diagram for simplifying:

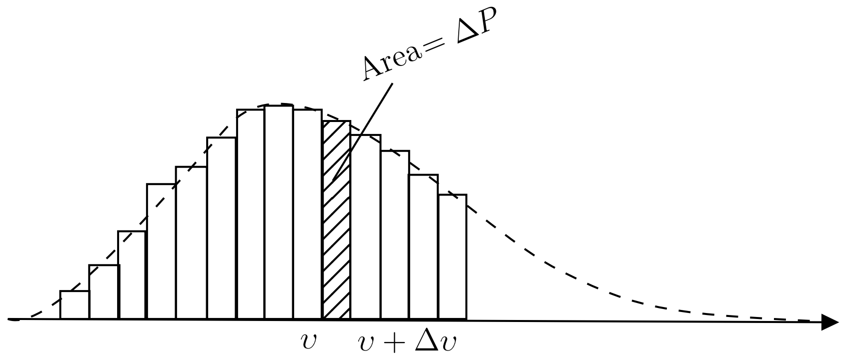


Figure 1: Visualation with histogram

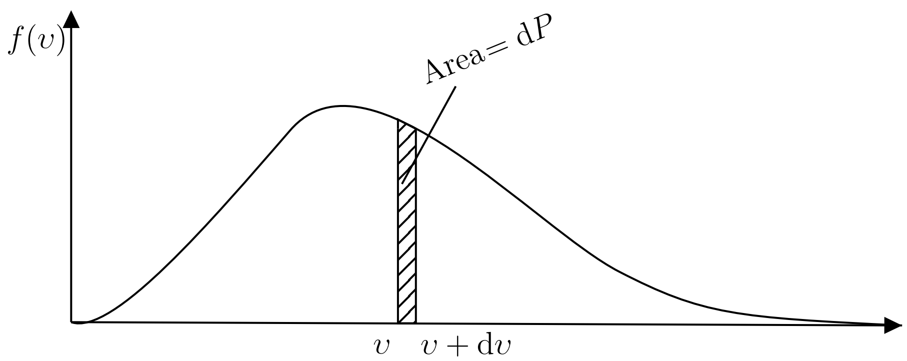


Figure 2: Visualation with graph

The histogram characterizes graphically the probability of obtaining results of measurements confined within different intervals of width Δv. If we take the limit $\Delta v \to 0$ we will get those histogram transforms to smooth curve.The function $f(v)$ defining this curve analytically is called a *probability distribution function.*

In accordance with the procedure followed in plotting the distribution curve, the area of the bar of width $\mathrm{d}v$ equals the probability of the fact that the result of a measurement will be within the range from v to $u + \mathrm{d}v$. Denoting this probability by $\mathrm{d}P$ we can write that

$$\mathrm{d}P(v) = f(v)\mathrm{d}v$$

So integrating both sides we see that

$$\int f(v)\mathrm{d}v = 1$$

Knowing the $f(v)$ we can find the mean values:

$$\langle v \rangle = \int v\mathrm{d}P = \int vf(v)\mathrm{d}v$$

Or with the similar method we can find mean value of any function $g(v)$

$$\langle g(v) \rangle = \int g(v)f(v)\mathrm{d}u$$

2 The Maxwell's Distribution

2.1 Proving the Maxwell's Distribution

We shall use the following procedure to find a way of quantitatively describing the distribution of molecules by velocity magnitudes. Let us take Cartesian coordinate axes in an imaginary space which we shall call v-space (velocity space). We shall lay off the values of v_x, v_y, v_z of individual molecules along these axes (what we have in view are the velocity components along the axes x, y, z taken in conventional space). Hence, a point in this v-space will correspond to the velocity of each molecule. Owing to collisions, the positions of the points will continuously change, but their density at each place will remain unchanged (we are dealing with equlibrium state).

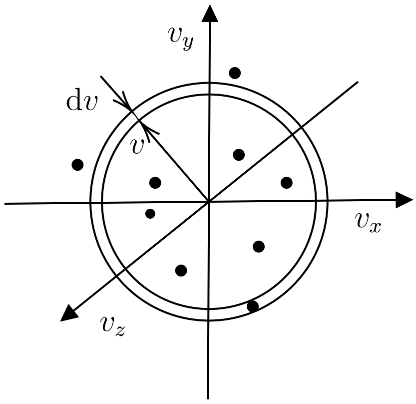


Figure 3: v-space

Owing to all the directions of motion having equal rights, the arrangement of the points relative to the origin of coordinates will be spherically symmetrical. Hence, the density of the points in our v-space can depend only on the magnitude of the velocity v. Let us denote this density by $Nf(v)$ (here N is the total number of molecules in the given mass of gas). Hence, the number of molecules whose velocity components are within the limits from v_x to $v_x + \mathrm{d}v_x$, from v_y to $v_y + \mathrm{d}v_y$, and from v_z to $v_z + \mathrm{d}v_z$ can be written in the form

$$\mathrm{d}N_{v_x,v_y,v_z} = Nf(v)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z \tag{2}$$

The product of three small changes gives ann element of volume in v-space.

So from the volume of the element in v-space the equation simplifies:

$$\mathrm{d}N_v = Nf(v)4\pi v^2\mathrm{d}v \tag{3}$$

the probability of the velocity component v_x of a molecule having a value within the limits from v_x to $v_x + \mathrm{d}v_x$ can be written in the form

$$\mathrm{d}P_{v_x} = \phi(v_x)\mathrm{d}v_x$$

where ϕ is distribution function. For the other components the equations will symmetrical. So by the equation 1:

$$\mathrm{d}P_{v_x,v_y,v_z} = \phi(v_x)\phi(v_y)\phi(v_z)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z$$

Also taking into account equation 2 we get that:

$$f(v) = \phi(v_x)\phi(v_y)\phi(v_z) \tag{4}$$

So if we take logarithm both sides :

$$\ln f(v) = \ln \phi(v_x) + \ln \phi(v_y) + \ln \phi(v_z)$$

differentiating this equation with respect to v_x:

$$\frac{f'(v)}{f(v)} \frac{\partial v}{\partial v_x} = \frac{\phi'(v_x)}{\phi(v_x)} \tag{5}$$

Since $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ we can take partial derivative

$$\frac{\partial v}{\partial v_x} = \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x}{v}$$

Plugging this in equation 5:

$$\frac{f'(v)}{f(v)} \frac{1}{v} = \frac{\phi'(v_x)}{\phi(v_x)} \frac{1}{v_x}$$

Since right hand side is not dependent on v_y and v_z(also left hand side too, it is an equality). Consequently it cannot depend on v_x too because they are symmetrical in definition of $f(v)$, namely equation 4.So that equation is equal to constant. Let that constant be $-\alpha$(why negative, it can be positive but at the end you will see it becames negative) So:

$$\frac{\phi'(v_x)}{\phi(v_x)} = -\alpha v_x$$

Integrating both sides:

$$\phi(v_x)) = C \exp\left(-\frac{\alpha v_x^2}{2}\right)$$

For v_y and v_z equations will be symmetrical. Thus by the equation 4:

$$f(v) = C^3 \exp\left(-\frac{\alpha(v_x^2 + v_y^2 + v_z^2)}{2}\right) = C^3 e^{-\frac{\alpha v^2}{2}}$$

We can find constant C from normalization of ϕ(note that v_x can be any real number):

$$C \int_{-\infty}^{\infty} \exp\left(-\frac{\alpha v_x^2}{2}\right) \mathrm{d}v_x = 1 \tag{6}$$

We can substitute new variable and change this to Gaussian integral. So at the end we get

$$C = \sqrt{\frac{\alpha}{2\pi}}$$

So our distribution functions:

$$\phi(v_x) = \sqrt{\frac{\alpha}{2\pi}} \exp\left(-\frac{\alpha v_x^2}{2}\right) \tag{7}$$

$$f(v) = \left(\frac{\alpha}{2\pi}\right)^{3/2} \exp\left(-\frac{\alpha v^2}{2}\right) \tag{8}$$

To find constant α we will calculate the value of $\langle v_x^2 \rangle$ with using equation 7 and knowing that it's equal to $\frac{kT}{m}$ according to **Lemma** 1. So:

$$\langle v_x^2 \rangle = \int_{-\infty}^{\infty} v_x^2 \phi(v_x) \mathrm{d}v_x = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{\alpha v_x^2}{2}\right) \mathrm{d}v_x \tag{9}$$

And this can be done with integration by parts and Gaussian integral. So overall we get:

$$\langle v_x^2 \rangle = \frac{1}{\alpha} \implies \alpha = \frac{m}{kT}$$

So finally we show that

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2kT}\right)$$

But if we want to find the actual probability distribution function, we have to multiply it by $4\pi v^2$ because of equation 3

So overall:

$$\boxed{F(v) = 4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2kT}\right)} \qquad \blacksquare$$

Lemma 1. *In the ideal gases the mean square of the one of the components of the velocity is equals to* $\dfrac{kT}{m}$

Proof. We need to relate pressure to energy so that we can get equation for root mean square velocity. Consider this scenario

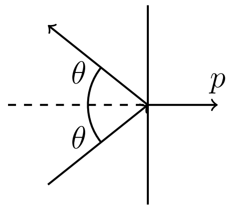


Figure 4: Molecule collides with the wall

So the momentum change is

$$\mathrm{d}p = 2mv\cos\theta \mathrm{d}n = \mathrm{d}N \frac{\mathrm{d}\Omega}{4\pi} \frac{2mv^2 \cos^2\theta \Delta S \Delta t}{V}$$

If we substitute formula for $\mathrm{d}\Omega$ and integrate it we get

$$\mathrm{d}p = \mathrm{d}N \frac{2mv^2 \Delta S \Delta t}{2\pi V} \underbrace{\int_0^{\pi/2} \cos^2\theta \sin\theta \mathrm{d}\theta}_{1/3} \underbrace{\int_0^{2\pi} \mathrm{d}\phi}_{2\pi}$$

So overall

$$\mathrm{d}p = \mathrm{d}N \frac{mv^2 \Delta S \Delta t}{3V}$$

Once again integrating this expression we get total momentum change in area ΔS with time Δt

$$\Delta p = \frac{m\Delta S \Delta t}{3V} \int_0^{\infty} v^2 \mathrm{d}N$$

The expression $\dfrac{1}{N}\displaystyle\int_0^{\infty} v^2 \mathrm{d}N$ is the mean value of square velocity. So substituting this yields

$$\Delta p = \frac{m\Delta S \Delta t}{3V} N \langle v^2 \rangle = \frac{1}{3} nm \langle v^2 \rangle \Delta S \Delta t$$

where n is molecules per volume. This is momentum change. So if we divide it by Δt we will get force, and again dividing by ΔS we will get the pressure.So

$$P = \frac{1}{3}nm\langle v^2\rangle = \frac{2}{3}n\left\langle\frac{mv^2}{2}\right\rangle = \frac{2}{3}m\langle\varepsilon\rangle$$

And if we compare this result with the ideal gas law we can see that

$$\langle\varepsilon\rangle = \frac{3}{2}kT$$

And by the definition of kinetic energy

$$\langle v^2\rangle = \frac{3kT}{m}$$

Also the velocity has components: $\langle v^2\rangle = \langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle$. In this equation all three components has equal rights, so they have to be equal. Overall we get that

$$\langle v_x^2\rangle = \frac{kT}{m}$$

■

2.2 Applying Maxwell's Distribution

We have the probability distribution function, so we can find some values like mean velocity or most probable velocity.
For finding mean velocity $\langle v\rangle$ we need to integrate:

$$\int_0^\infty v^2F(v)\mathrm{d}v = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_0^\infty \exp\left(-\frac{mv^2}{2kT}\right)v^3\mathrm{d}v \tag{10}$$

So we can integrate this with substitution and integration by parts.We get:

$$\langle v\rangle = \sqrt{\frac{8kT}{\pi m}}$$

With the same method we can also find $\langle v^2\rangle$. So integrating:

$$\langle v^2\rangle = \int_0^\infty v^2F(v)\mathrm{d}v = \frac{3kT}{m}$$

And the root of this velocity is called root mean square velocity:

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

For the most probable velocity we need to find the maximum of the $F(v)$, which can be found by taking derivative and setting equal to zero:

$$\exp\left(-\frac{mv^2}{2kT}\right)\left(2-\frac{mv^2}{kT}\right) = 0$$

From here it is obvious that

$$v_{mp} = \sqrt{\frac{2kT}{m}}$$

2.3 Maxwell's Distribution in spherical coordinates

We can even go further and write Maxwell's distribution not only for a sphere shell but also a tiny element on the system. It is more convenient to use spherical coordinates in our v-space. Changing from cartesian to spherical:

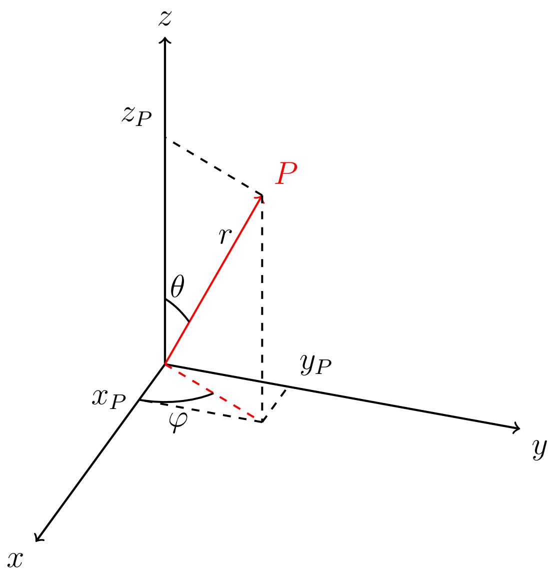


Figure 5: Spherical coordinate system

The little volume in the cartesian coordinates is $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ But we want to simplify things so from basic trigonometry it is obvious that

$$x_P r \sin\theta\cos\varphi \quad y_P = r\sin\theta\sin\varphi \quad z_P = r\cos\theta$$

So tiny volume in the spherical coordinates is

$$\mathrm{d}V = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}r \mathrm{d}\varphi$$

So Maxwell's distribution changes into(in our v-space $r = v$):

$$f(v) = v^2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2kT}\right) \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$$

So we can use this when averaging things, due to coordinates.

3 Problems

3.1 Easy problems

Problem 1
Do the integrals provided in Section 2.

From now on, you can use following integrals:

$$I_0 = \int_0^\infty e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{11}$$

$$I_1 = \int_0^\infty xe^{-ax^2} = \frac{1}{2a} \tag{12}$$

$$I_2 = \int_0^\infty x^2e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \tag{13}$$

$$I_3 = \int_0^\infty x^3e^{-ax^2} = \frac{1}{2a^2} \tag{14}$$

$$I_4 = \int_0^\infty x^4e^{-ax^2} = \frac{3}{8}\sqrt{\frac{\pi}{a^5}} \tag{15}$$

$$I_5 = \int_0^\infty x^5e^{-ax^2} = \frac{1}{a^3} \tag{16}$$

Problem 2
Show that for any ideal gas the product of mean velocity and the mean inverse velocity is:

$$\langle v \rangle \left\langle \frac{1}{v} \right\rangle = \frac{4}{\pi}$$

Problem 3
If the formula for Maxwell's distribution in terms of kinetic energy E is given as:

$$F(E)\mathrm{d}E = aE^b \exp\left(-\frac{E}{kT}\right)\mathrm{d}E$$

Find the constants a and b.

Problem 4
We can also define the distribution in terms of the De-Broglie wavelength:

$$\lambda = \frac{h}{mv}$$

where h is Planck constant. Suppose that it is given as

$$F(\lambda)\mathrm{d}\lambda = a\lambda^{-b} \exp\left(-\frac{c}{\lambda^2}\right)\mathrm{d}\lambda$$

then find constants a, b and c.

3.2 Intermediate problems

Problem 5

One mole of ideal gas kept in the vessel. Temperature of the gas is kept constant equal to T. Gas concentration in the vessel is n. Estimate the number of molecules N_c colliging with the flat wall of container per unit area S during period of time Δt.
Hint: First try a simplified problem then try to generalize your result.

Problem 6

Using Maxwell's distribution calculate pressure of P of ideal gas with concentration n and temperature T.

3.3 Harder problems

Problem 7

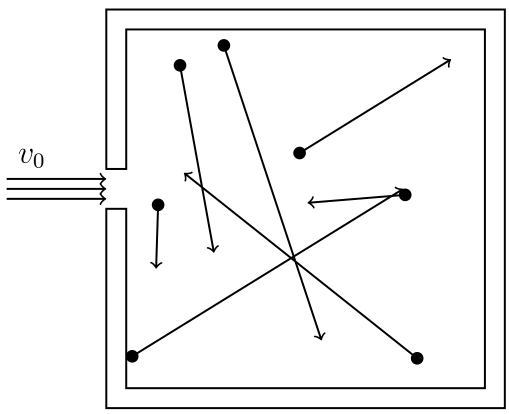
An ideal monoatomic gas is leaking from thermally insulated vessel into vacuum through a tiny hole ,which is much smaller than the mean free path of gas. Assuming Maxwell's velocity distribution for the atoms, calculate the parameter γ, which is defined as the ratio between average kinetic energy of molecules of gas outside the vessel to average energy oh the gas inside the vessel:

$$\gamma = \frac{\langle E_{\text{outside}} \rangle}{\langle E_{\text{inside}} \rangle}$$

Assume that none of the atoms flow back.

Problem 8

A thermally insulated vessel with thin walls has a small hole at one of its sides. This vessel was initially empty at the vacuum. A thin beam of molecules moving with equal velocities v_0 is directed at the hole of the vessel in a direction perpendicular to the surface of the hole. (The arrows show gas molecules.)



Determine the temperature of the gas T inside the vessel after a long period of time. Molar mass of the gas is μ.

3.4 Hardest Problems

Try these problems:

- 2020 OPhO Invitational Round Problem 7
- 2002 APhO Problem 3
- 2015 APhO Problem 2(b)
- 2019 APhO Problem 2B

4 Solutions

Solution 1

1. So first integral is equation 6. To do that we just need to substitute $u^2 = \dfrac{\alpha v^2}{2}$ and $\mathrm{d}u = \mathrm{d}v\sqrt{\dfrac{\alpha}{2}}$. So integral becomes

$$C\sqrt{\frac{2}{\alpha}}\int_{-\infty}^{\infty} e^{-u^2}\mathrm{d}u = 1.$$

And this is Gaussian integral which is equals to $\sqrt{\pi}$. So constant is

$$\sqrt{\frac{\alpha}{2\pi}}$$

2. Second integral is equation 9.

$$\langle v^2\rangle = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{\alpha v_x^2}{2}\right)\mathrm{d}v_x$$

This can be doable using integration by parts. So we want to differentiate v_x and integrate other part

$$\begin{array}{ccc} & \text{D} & \text{I} \\ + & v_x & v_x e^{-\frac{\alpha v_x^2}{2}} \\ - & 1 & -\dfrac{1}{\alpha}e^{-\frac{\alpha v_x^2}{2}} \end{array}$$

So integral becomes

$$-\frac{v_x}{2\alpha}e^{-\frac{\alpha v_x^2}{2}}\Big|_{-\infty}^{\infty} + \frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-\frac{\alpha v_x^2}{2}}\mathrm{d}v_x$$

First term vanishes because of exponential. Second term is same as integral 1. So overall doing same steps we get

$$\langle v_x^2\rangle = \sqrt{\frac{\alpha}{2\pi}}\sqrt{\frac{2\pi}{\alpha^3}} = \frac{1}{\alpha}$$

3. Third integral is equation 10. We can use same method for this.but this time we will different v^2. So

$$\begin{array}{ccc} & \text{D} & \text{I} \\ + & v^2 & v\exp\Big(-\dfrac{mv^2}{2kT}\Big) \\ - & 2v & -\dfrac{kT}{m}\exp\Big(-\dfrac{mv^2}{2kT}\Big) \end{array}$$

The same as integral two first term vanishes because of exponential and we left with

$$\int_0^\infty \frac{kT}{m}2v\exp\Big(-\frac{mv^2}{kT}\Big)\mathrm{d}v$$

So if we substitute $u = \dfrac{mv^2}{kT}$ we get $\displaystyle\int_0^\infty 2\Big(\frac{kT}{m}\Big)^2 e^{-u}\mathrm{d}u = 2\Big(\frac{kT}{m}\Big)^2$ Overall

$$\langle v\rangle = 4\pi\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}2\Big(\frac{kT}{m}\Big)^2 = \sqrt{\frac{8kT}{\pi m}}$$

Solution 2

For this problem we just need to find $\left\langle\dfrac{1}{v}\right\rangle$ So by the definition

$$\left\langle\frac{1}{v}\right\rangle = \int_0^\infty \frac{f(v)}{v}\mathrm{d}v = 4\pi\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\int_0^\infty v\exp\Big(-\frac{mv^2}{2kT}\Big)\mathrm{d}v$$

using provided integral 12 we get that

$$\left\langle\frac{1}{v}\right\rangle = 4\pi\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\frac{kT}{m} = \sqrt{\frac{2m}{\pi kT}}$$

And $\langle v\rangle = \sqrt{\dfrac{8kT}{\pi m}}$ is known so if we multiply them we get $\dfrac{4}{\pi}$

Solution 3

Original Maxwell's distribution is

$$f(v)\mathrm{d}v = 4\pi v^2\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\exp\Big(-\frac{mv^2}{2kT}\Big)\mathrm{d}v$$

For transformation we must have

$$F(E)\mathrm{d}E = f(v)\mathrm{d}v \implies F(E)\mathrm{d}E = f(v)\frac{\mathrm{d}v}{\mathrm{d}E}\mathrm{d}E$$

We can find its derivative by the definition $E = \dfrac{mv^2}{2} \implies \dfrac{\mathrm{d}E}{\mathrm{d}v} = mv$ So $F(E)\mathrm{d}E$ is

$$F(E)\mathrm{d}E = 4\pi v^2\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\exp\Big(-\frac{mv^2}{2kT}\Big)\Big(\frac{1}{mv}\Big)\mathrm{d}E =$$
$$= \frac{2}{\sqrt{\pi}(kT)^{3/2}}\sqrt{E}\exp\Big(-\frac{E}{kT}\Big)\mathrm{d}E$$

Or

$$a = \frac{2}{\sqrt{\pi}(kT)^{3/2}} \qquad b = \frac{1}{2}$$

Solution 4
New deBroglie wavelength distribution should satisfy the condition

$$f(v)\mathrm{d}v = -F(\lambda)\mathrm{d}\lambda \implies -f(v)\frac{\mathrm{d}v}{\mathrm{d}\lambda}\mathrm{d}\lambda$$

By definition $\frac{\mathrm{d}\lambda}{\mathrm{d}v} = -\frac{h}{mv^2}$ Combining equations and replacing parameter $v = \frac{h}{m\lambda}$ yields

$$F(\lambda)\mathrm{d}\lambda = \sqrt{\frac{2}{\pi}}\frac{h^3}{\lambda^4(mkT)^{3/2}}\exp\Big(-\frac{h^2}{2mkT\lambda^2}\Big)\mathrm{d}\lambda$$

or

$$a = \sqrt{\frac{2}{\pi}}\frac{h^3}{(mkT)^{3/2}} \quad b = 4 \quad c = \frac{h^2}{2mkT}$$

Solution 5
Like hint says lets first look at simplified problem. Lets assume all the molecules from origin move with velocity v in cylindrical shape and to one direction θ with the normal of the wall
Stage 1
Then during time interval Δt only molecules with height $v\Delta t$ can strike to the wall.The number of molecules can strike will be

$$N_1 = n\Delta V = nv\Delta t S\cos\theta$$

Where $S\cos\theta$ is the cross sectional area of that beam.
Stage 2
Now lets average this with Maxwell's distribution. The range of velocity is $[0;\infty]$ the polar angle $[0;2\pi]$ the azimithual angle is $[0;\frac{\pi}{2}]$. Now

$$\begin{aligned}N_c &= \sum nSv\cos\theta\Delta\Big(v^2\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\exp\Big(-\frac{mv^2}{2kT}\Big)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}v\Big)\\ &= nS\Delta t\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\int_0^\infty v^3\exp\Big(-\frac{mv^2}{2kT}\Big)\mathrm{d}v\int_0^{\pi/2}\cos\theta\sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\end{aligned}$$

Using provided integral 14

$$N_c = nS\Delta t\Big(\frac{m}{2\pi kT}\Big)^{\frac{3}{2}}\cdot\Big(\frac{1}{2}\frac{(2kT)^2}{m^2}\Big)\frac{1}{2}\cdot 2\pi$$

So canceling out terms we get an accurate estimation for number of collisions for ideal gas with temperature T is

$$N_c = n\sqrt{\frac{kT}{2\pi m}}S\Delta t$$

Solution 6
So we can consider the scenario in the figure 4 for simplified problem. So we know that the pressure will be $P = 2mnv^2\cos^2\theta$ from proof the lemma 1.

Let's average it with Maxwell's distribution.

$$P = \sum 2mnv^2\cos^2\theta \cdot \left(v^2\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left(-\frac{mv^2}{2kT}\right)\sin\theta \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}v\right)$$
$$= 2mn\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\int_0^\infty v^4\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v\int_0^{\pi/2}\cos^2\theta\sin\theta \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi$$

Here we can use the integral 15 and get the result:

$$P = 2mn\left(\frac{m}{2\pi kT}\right)^{3/2}\cdot\left(\frac{3\sqrt{\pi}}{8}\left(\frac{5kT}{m}\right)^{5/2}\right)\cdot\frac{1}{3}2\pi = nkT$$

Solution 7

Average kinetic energy inside of container is

$$\langle E_{\text{inside}}\rangle = \frac{3}{2}kT$$

according to lemma 1 or you can calculate it by

$$\langle E_{\text{inside}}\rangle = \int_0^\infty \frac{mv^2}{2}f(v)\mathrm{d}v$$

For the outside of vessel we can calculate it like other problems. Consider a simple scenario with all the molecules with the same velocity they are mocing in same direction. In this case the number of molecules that go outside will be

$$N_{\text{out}} = nvS\cos\theta\Delta t$$

Now let's average this energy and number of molecules with Maxwell's distribution. We have derived number of molecules before. It is

$$N_{\text{out}} = n\sqrt{\frac{kT}{2\pi m}}S\Delta t$$

So applying similar approach for E_{out}

$$E_{\text{out}} = \sum \frac{mv^2}{2}nvS\cos\theta\Delta t\left(u^2\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left(-\frac{mv^2}{2kT}\right)\sin\theta \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}v\right)$$
$$= \frac{mnS\Delta t}{2}\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^5\exp\left(-\frac{mv^2}{2kT}\right)\mathrm{d}v\int_0^{\pi/2}\cos\theta\sin\theta \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi$$

Using the integral 16 yields

$$E_{\text{out}} = nS\Delta t 2kT\sqrt{\frac{kT}{2\pi m}}$$

The avarage energy of molecules in the outside will be

$$\langle E_{\text{outside}}\rangle = \frac{E_{\text{out}}}{N_{\text{out}}}$$

So dividing energy to number of molecules we get

$$\langle E_{\text{out}}\rangle = 2kT$$

Our final result is

$$\frac{\langle E_{\text{outside}}\rangle}{\langle E_{\text{inside}}\rangle} = \frac{4}{3}$$

Note
For the solution of OPhO problem 7 check this Aops forum