Parabolic Projectiles

Optimum case problems

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The general idea is that the equation of the trajectory of a projectile is given by,

$$y = x \tan \theta_0 - \frac{gx^2}{2(v_0 \cos \theta)^2}$$

Which certifies that the trajectory is strictly a "Parabola". And thus, the parabolic trajectory must have a "Focus". Again, Parabola has some fundamental definition of being the "locus" of the points that are at equal distance from a line called *Directrix* and a focus point F.

Problem 1 (Focus of the Projectile)**.** For a projectile with initial speed v_0, find the distance of the focus point F from the launching point O.

There are two method of doing the problem, one is invoking the geometry at the launching point and relating it to the vertical line and focus point, another method is to use the definition of parabola using the directrix. Both are interesting, but I prefer geometric one as it seems to be more powerful in terms of intuition.

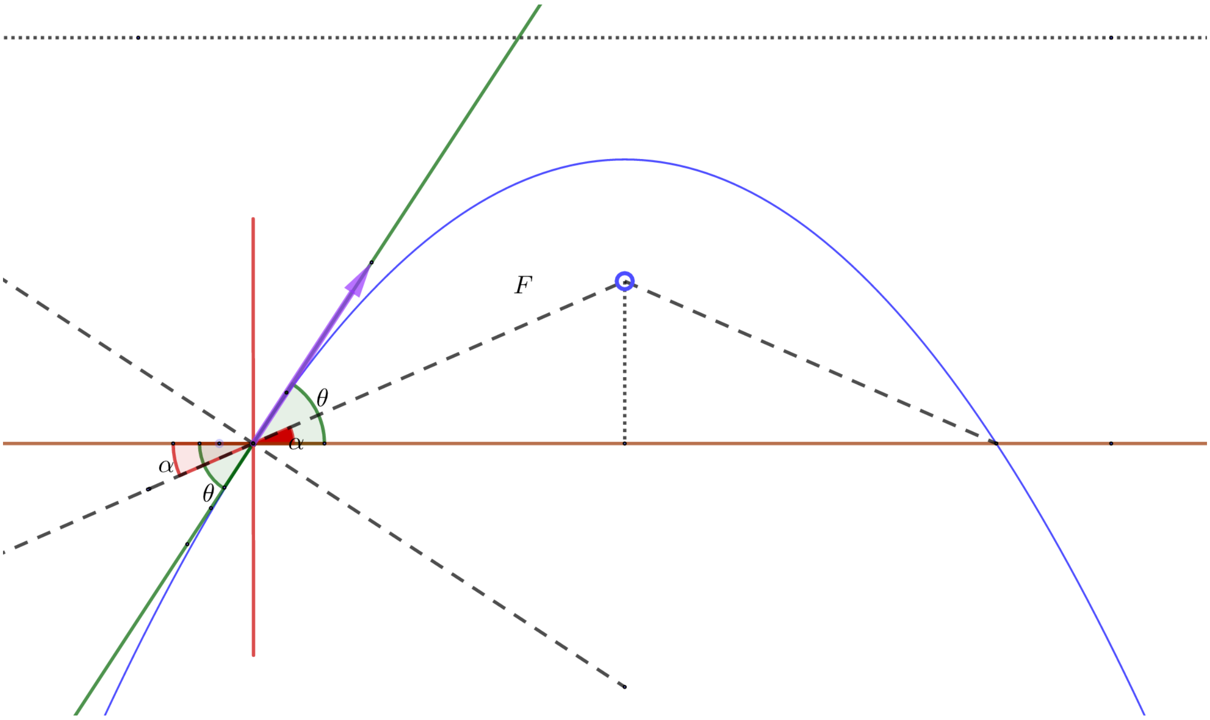

Figure 0.0.1

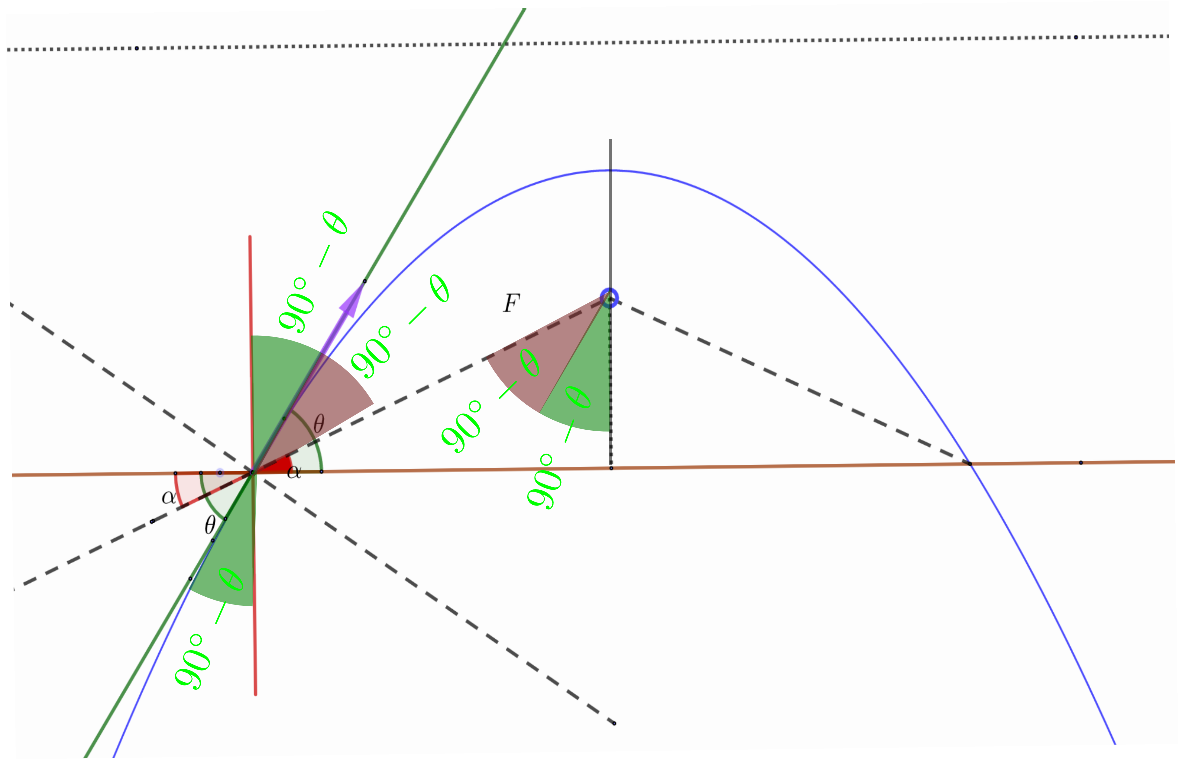

Figure 0.0.2

Solution. We launch the mass at θ angle, at speed v_0. Now, there exists a triangle with corners launching point, focus, and vertically below focus. The end of triangle is called α. Now looking at the dotted line that is perpendicular to the initial velocity reminds us that parabolic mirror focuses all the light to the focus. Thus an incoming light, vertically coming towards the mirror wil be reflected towards the focus.
Here the red vertical line crossing point O should reflect to focal point F if it was a mirror. By law of reflection, the angle made by red line with the dotted line (perp to $\vec{v}_0$) is equal to the dotted line with OF line. Using this we can come to the second diagram of this problem. There, the dark green angle region in the lower left is equal to $90° - \theta$. This is made possible with the law of reflection rule we made.
And this dark green region has a opposite angle. But, this has a neighbour dark magenta that is also $90° - \theta$. Together they make an alternate angle at the focus point region as in the diagram.
Finally, using the triangle consisting focus, O tells us that,

$$180° = \alpha + 90° + 180° - 2\theta$$

This leaves us with,

$$\alpha = 2\theta - 90°$$

And the other angle with focus is,

$$180° - 2\theta$$

This is clear that the orthogonal projection of the line OF is half of the range of projectile. The range is,

$$R = \frac{2v_0^2 \sin\theta \cos\theta}{g}$$

Half of the projectile with necessary simplification,

$$\frac{R}{2} = \frac{v_0^2 \sin 2\theta}{2g}$$

Now,

$$OF \ \sin(180° - 2\theta) = OF \ \sin 2\theta = \frac{v_0^2 \sin 2\theta}{2g}$$

So, we finally find that ,

$$OF = \frac{v_0^2}{2g}$$

Theorectical Theorem 1 (Focus of Projectile) — The Focus F of a projectile is located,

$$r = \frac{v_0^2}{2g}$$

from the launching point at an angle,

$$\alpha = 2\theta - 90°$$

> if launching angle is θ. Also note that the horizontal projection of the r line is half of the "Projectile Range". So, this also helps finding the range of the projectile.

The distance is not dependent on θ, this make the line a kind of vector $\vec{r}$ which can rotate with varied launching angle θ, without affecting the line length. We also note that projection of the $\vec{r}$ is the half of range. But when will the vector $\vec{r}$ make maximum projection on the ground?

It would when the whole vector is parallel to the ground, or I like to say, lying on the ground. Then the $\alpha = 0$, this tells us,

$$0 = 2\theta - 90^\circ \qquad \rightarrow \qquad \theta = 45^\circ$$

Which actually is the condition for a projectile range to be maximum!

> **Theorectical Theorem 2** (Range from Focus line) — The range of a projectile is maximum along the focus line. (Needs more attention to make sense).

We are at the point to actually understand what is a parabola.

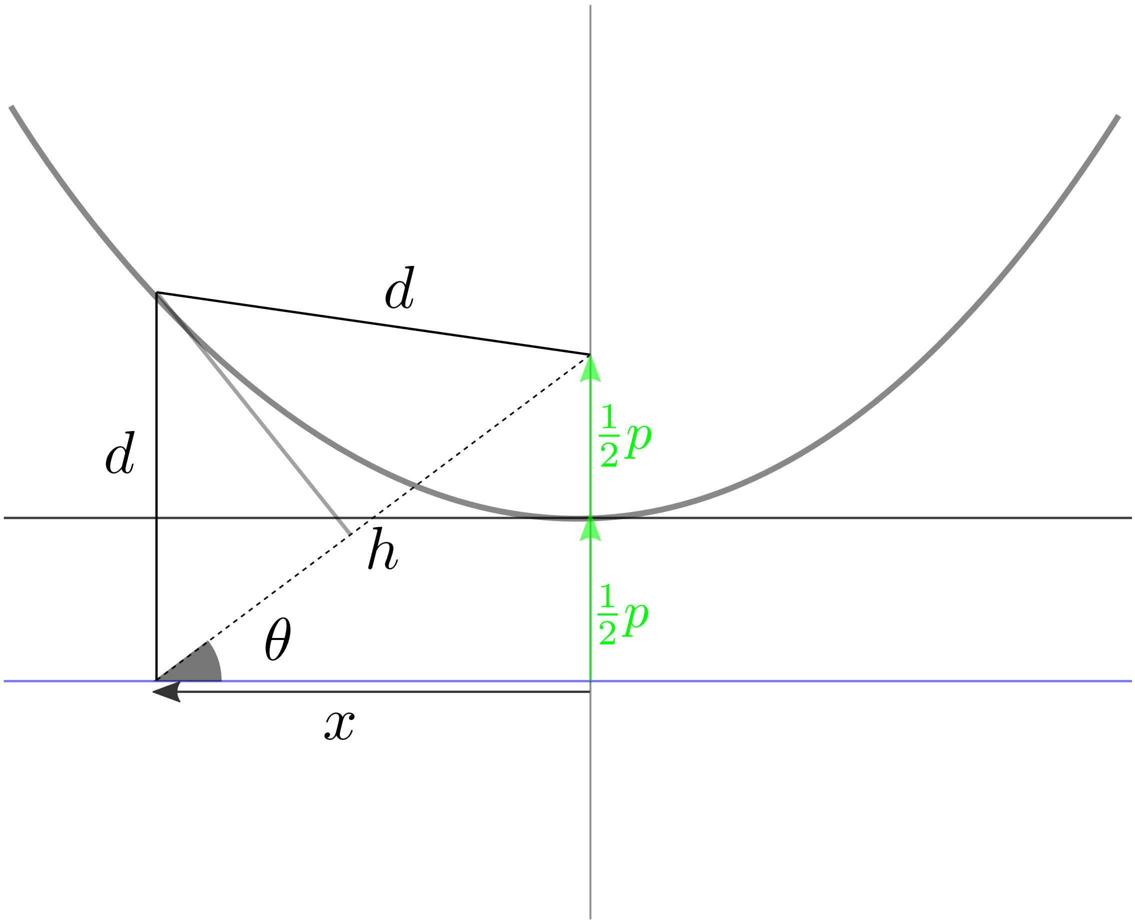

Figure 0.0.3

The lower bluish line is the so called "Directrix line". The focus is selected at any point at p height. Now, there exists a point that is d distance from the directrix and also d distance from the focus. This point lies in the parabola. Tracing out all the points that have this equality of distance gives us a parabola. It happens to be that the lowest point is $\frac{1}{2}p$ above directrix, and as rule says, $\frac{1}{2}p$ from focus too.

Can we find a rule to relate d and x as a function? We can try.

Notice that d builds a isosceles triangle, where the h is line joining the x on directrix and focus. We will say that $y = 0$ at the directrix line. So,

$$h^2 = x^2 + p^2$$

But the triangle invokes,

$$2d\cos(90^\circ - \theta) = 2d\sin\theta = h$$

But $\sin\theta = \frac{p}{h}$, so,

$$2d\frac{p}{h} = h$$

And,

$$2dp = x^2 + p^2$$

We have found a required equation, but let us change our frame's height, so that the minima of the parabola is located at (0,0), so,

$$y = d - \frac{1}{2}p \qquad \rightarrow \qquad d = y + \frac{1}{2}p$$

So,

$$2\left(y + \frac{1}{2}p\right)p = x^2 + p^2$$

Giving us,

$$x^2 = 2py$$

So, all parabolic equation has,

$$y = \frac{x^2}{2p}$$

form, recalling p is distance of directrix from focus.

Problem 2 (`Again, with Locus rule`)**.** For a projectile with initial speed v_0, find the distance of the focus point F from the launching point O. Now use the locus rule of parabola we just derived.

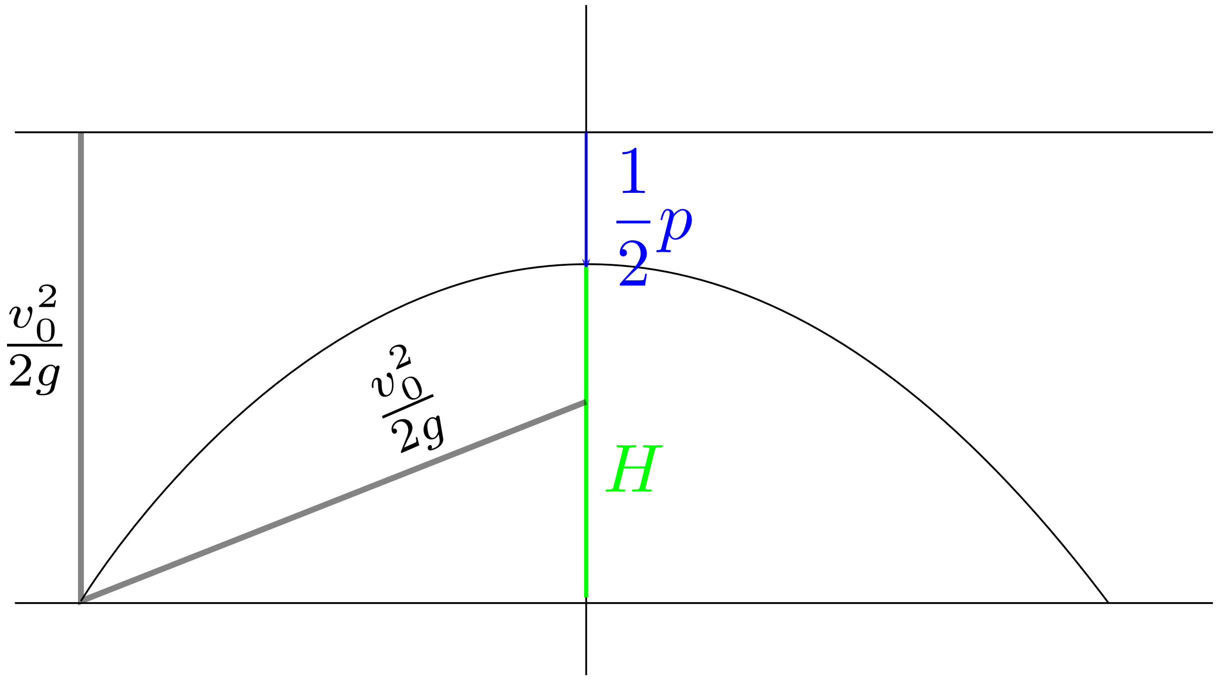

Figure 0.0.4: The distance of the Directrix from the Focus is p. And thus, we can easily notice that distance between max altitude of projectile and Directrix is $\frac{1}{2}p$.

Solution. The max height possible,

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

We can see that the distance of focus is going to be $\frac{1}{2}p + H$ from the launching point. The half of range of the projectile,

$$\frac{R}{2} = \frac{v_0^2 \sin\theta\cos\theta}{g}$$

As we know,

$$\frac{x^2}{2p} = y$$

Looking at the left vertical bar, making our head upside down, with apparently $y = 0$ as the tip H above, then $R/2$ aside, the height of parabola is going to be H. Using this,

$$\left(\frac{v_0^2 \sin\theta\cos\theta}{g}\right)\frac{1}{2p} = \frac{v_0^2 \sin^2\theta}{2g}$$

We found,

$$\frac{1}{2}p = \frac{v_0^2 \cos^2\theta}{2g}$$

Hence,

$$d = H + \frac{1}{2}p = \frac{v_0^2 \sin^2\theta}{2g} + \frac{v_0^2 \cos^2\theta}{2g} = \frac{v_0^2}{2g}$$

So, we have got another type of solution,

$$\boxed{r = \frac{v_0^2}{2g}}$$

Problem 3 (Above the roof!). There is this roof with dimension in the figure, taken from J. Kalda's Mech, and we are to find the least possible speed to topple over the roof thrown from anywhere left, from the ground.

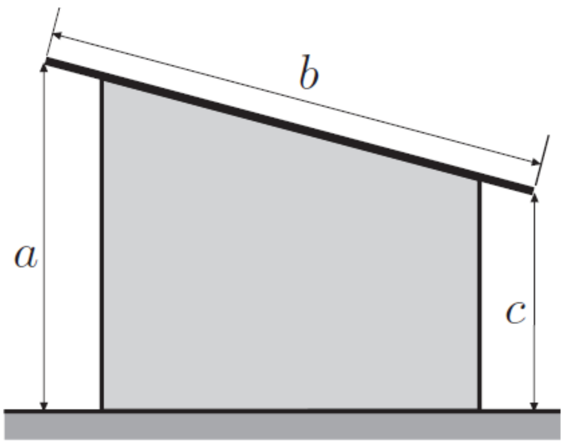

Figure 0.0.5

Solution. The projection of the $\vec{r}$ focus vector is the range of the particle in air. If the projectile has the focus vector $\vec{r}$ such a way that it lies on the required surface (the inclined roof), the projection is max and thus the maximum distance is being covered for the speed, and if this equals to the roof dimensions, we have the optimal case.
Let at the left corner of the roof, the speed of the particle be v, and we know, if the speed it is launched from ground is v_0

$$v^2 = v_0^2 - 2ag$$

Then the focus vector line is,

$$\frac{v^2}{2g} = \frac{v_0^2 - 2ag}{2g}$$

And when the particle falls and touches the right corner, the speed is,

$$v_{right}^2 = v_0^2 + 2g(a - c)$$

When falling, the v_{right} also has a focus line, this will meet the focus line of the left corner speed focus line given the trajectory is optimal. Refer to the diagram I made in paint.
The total length of the lines is going to be b, and thus,

$$\frac{v^2}{2g} + \frac{v^2 + 2g(a - c)}{2g} = b$$

Now plug that $v^2 = v_0^2 - 2ag$, and we solve this for v_0 and get,

$$\boxed{v_0^2 = g(a + b + c)}$$

I don't know why this depends on the perimeter of the roof, might be some interesting thing happening in the background!

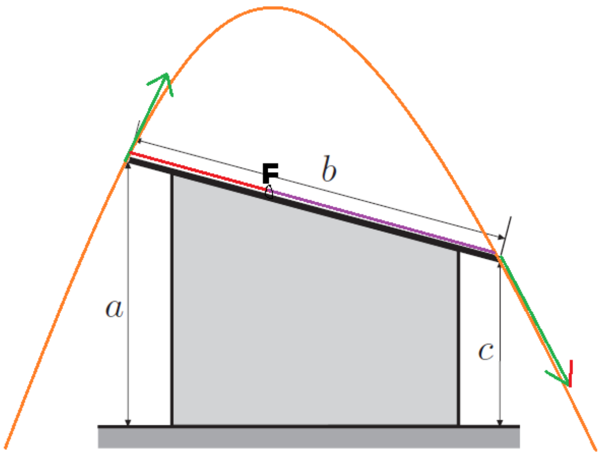

Figure 0.0.6

Problem 4 (`Physicist Grasshopper`). A wooden log of radius R lies in front of a Physicist Grasshoper back to home from college. The grasshoper calculates the minimum speed he needs to hop and get to the other point of the log and successfully does the stunt. What should be his minimal speed to do this?

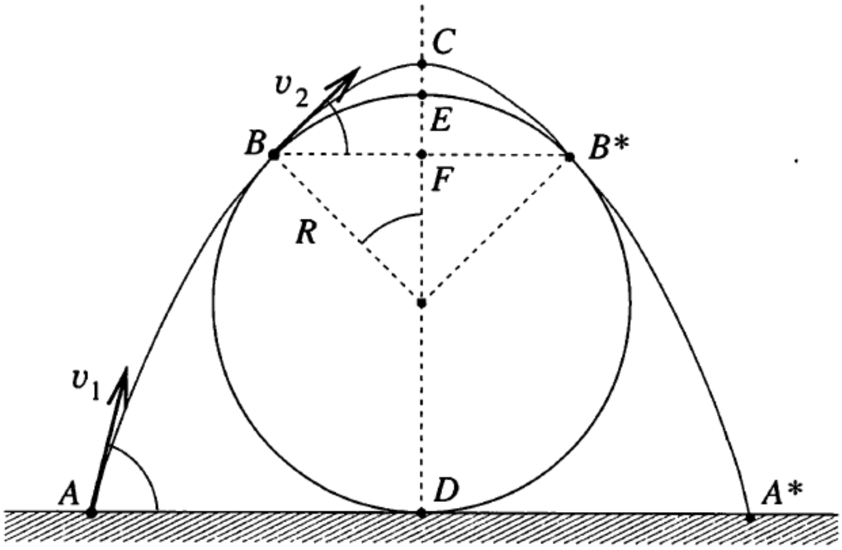

Figure 0.0.7

Solution. When trajectory is optimal, it surely touches the log at one point. When it does so, then that speed should make the max range. That will cause the grasshopper to reach other side with least speed with max possible distance. And because the line made by the touching points BB^* on log is horizontal, of course the velocity vector will make 45° with the horizontal.

$$v_2^2 = v_1^2 - 2gR\,(1 + \sin 45) = v_1^2 - 2gR\left(1 + \frac{1}{\sqrt{2}}\right) = v_1^2 - gR(2 - \sqrt{2})$$

And for the range, we can think that the focus line is horizontal, thus,

$$\frac{v_2^2}{2g} = 2R\cos 45^\circ = \sqrt{2}R$$

This tells us that,

$$v_1^2 = \sqrt{2}gR + gR\left(2 + \sqrt{2}\right)$$

So, the final answer is,

$$\boxed{v_0^2 = 2gR(1 + \sqrt{2})}$$

Problem 5 (Problem for some revision)**.** There is a tower of height h, from above it, balls are thrown in random directions but with constant speed v_0. Find the maximum range possible, if range is measured from the foot of tower to the landing point of projectile.

Answer: $d_{max} = \frac{v_0}{g}\sqrt{v_0^2 + 2gh}$, can be easily found if we imagine that the "Focus line" lies on the line drawn from the launching point to the landing point. This is because, the range is projection of the "Focus line", hence if the line it self is parallel to the starting and finishing points, that is ought to be the maximum range.

Note. The problem requires to find the *Maximum horizontal range*, not the optimum angle for maximum range. Many when I asked to help me with this problem just took brute force for finding θ, but that turns out to be harder than required. If you are looking for θ, then it is better to find range first then go for θ.

Reference:

1. *200 more Puzzling Physics Problems*
2. *200 Puzzling Physics Problems*
3. *IPHO 2012*
4. *Kinematics*, "Study Guides for IPhO" by Professor Jaan Kalda
5. *EuPhO 2019*