Solutions to

F=ma Mock Exam 2022

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1 Problem 1

Originally, there is only one set of springs of constant k, meaning a weight W would result in a compression of W/k. After adding the new set of springs, the effective spring constant is

$$\frac{1}{k_{\text{eff}}} = \frac{1}{k} + \frac{1}{2k} \implies k_{\text{eff}} = \frac{2k}{3}$$

which means a weight W_0 would result in a total compression of $3W_0/(2k)$. Therefore:

$$\frac{W'}{k} = \frac{3W_0}{2k}$$
$$\frac{W'}{W_0} = \frac{3}{2} \implies \boxed{B}$$

2 Problem 2

Note that the string has no way of dissipating energy, so the velocity remains at a constant v. Between consecutive collisions of the string with each corner, the length of the string is constant, so the mass undergoes uniform circular motion. Therefore, the total time is the sum of times taken to traverse each circular arc, which is

$$t = \frac{7\pi a}{2v} + \frac{6\pi a}{2v} + \cdots + \frac{2\pi a}{2v} + \frac{\pi a}{2v} = \frac{14\pi a}{v} \implies \boxed{B}$$

3 Problem 3

The maximum height is achieved by the hole in which the water shoots straight up. If the initial speed of water through the hole is v, then by continuity, the new speed v' of water going through a singular hole is

$$Nv = v'$$

Since $h \propto v^2$, $h' = N^2 h \implies \boxed{D}$

4 Problem 4

Let us consider the rotating reference frame of the four-way pipe. When the water initially leaves, it has a velocity $\mathbf{v}$ that is parallel to one of the pipes. Afterwards, it is affected by a coriolis acceleration $-2\boldsymbol{\omega} \times \mathbf{v}$ that points in the opposite direction of rotation. Note that the shape of the water relative to the pipes should be the same in all reference frames. Therefore, the only choice that satisfies both the parallel exit velocity and coiling in the opposite direction of the rotation is $\boxed{E}$.

5 Problem 5

The amount of energy required to take out a cubical crater of length L is proportional to

$$E \propto mgL \propto \rho L^3 gL$$

A meteor with twice the mass will be able to provide twice the energy. Therefore, the linear size of the crater will be $2^{1/4}$ larger, and its volume will be $2^{3/4}$ times larger, implying $\boxed{D}$.

6 Problem 6

The weight of the cylinder is negligible, so the total normal force of the ground on the cylinder is 10 N. By symmetry, the normal pressure is equal at all contact points and the frictional force is similarly equal in magnitude at all points, pointing in the opposite direction to the cylinder's rotation. Therefore, summing up all the frictional works of each point, the total rate at which heat is being dissipated is

$$P = \mu N v = \mu N \omega R$$

Therefore

$$\Delta Q = Pt = 0.6t \implies \boxed{A}$$

7 Problem 7

The tower will be stable if its center of mass is located within its area of support. If we set the origin of our coordinate system to be the bottom left corner, that's equivalent to saying

$$x_{\text{cm}} \leq b$$

Calculating x_{cm}:

$$x_{\text{cm}} = \frac{b/2 + b/2 + a/2}{3} = \frac{2b + a}{6}$$

Yields:

$$2b + a \leq 6b$$
$$a/b \leq 4 \implies \boxed{C}$$

8 Problem 8

Suppose the rod had an angle ϕ with the horizontal. In a time t, the bead would travel a distance

$$d = \frac{1}{2}at^2 = \frac{1}{2}gt^2 \sin\phi$$

along the rod. Now consider the set of points that can be reached from P in a time t. That set of points has a radial distance $r \propto \sin\phi$, which you may recognize as the polar form of a circle! In particular, the set of points reachable in time t is a circle with P as its uppermost point.

From this information, we can determine that the shortest time is reached if the rod is placed at the tangency point of two circles: the circle centered at the origin, and the circle whose upper most point is P and is tangent to the circle centered at the origin. Using geometry, we can determine that

$$\theta \approx 53^\circ \implies \boxed{B}$$

9 Problem 9

Suppose the small hole has area A. Just before breaking, the tape experiences a force

$$6 \text{ N} = PA = (\rho g h)A$$

After breaking, water exits at a velocity of $\sqrt{2gh}$. By momentum conservation, the force that the exiting water exerts on the remaining water is

$$F = \frac{dp}{dt} = \frac{dm}{dt}v = \rho A v^2 = 2\rho g h A = 12 \text{ N} \implies \boxed{D}$$

10 Problem 10

By conservation of angular momentum,

$$I_{\text{cm}}\omega_0 = I\omega$$

It is well known that $I_{\text{cm}} = MR^2/2$, and by the parallel axis theorem

$$I = I_{\text{cm}} + M(R/3)^2 = \frac{11}{18}MR^2$$

Therefore:

$$\omega = \frac{9}{11}\omega_0 \implies \boxed{E}$$

11 Problem 11

Suppose the low density solid we are trying to measure has a volume V. Let us call the density of the gas ρ_g. The true weight of the solid is $\rho_1 V$, but the scale results the measurement of a weight

$$(\rho_1 - \rho_g)V$$

Therefore, the relative error is

$$\frac{\rho_g}{\rho_1} = 2\% \implies \boxed{E}$$

12 Problem 12

The ladder experiences 5 forces: two normal forces, two frictional forces, and the weight of the construction worker. However, for the maximum height not to slip, the frictional forces will be maximized to μN, so in fact we can combine each of the friction+normal pairs to get 3 effective forces. Finally, each of the lines of the "3" forces, if extended, must intersect at a single point in order for torque to be balanced, because otherwise we can take torque about the point of intersection of two forces and find a non-zero torque.

Let us find the position of the construction worker in this scenario. Suppose the ladder is located in the first quadrant, and the x and y axes are the ground and wall respectively. The lines that the forces from the wall and the ground make are

$$y = L\sin\theta + \mu x$$
$$y = -\frac{1}{\mu}(x - L\cos\theta)$$

respectively. These lines intersect at an x value of

$$x = \frac{L(\cos\theta - \mu\sin\theta)}{1+\mu^2}$$

which corresponds to a y value on the ladder of

$$y = L\sin\theta - x\tan\theta = 0.15 \text{ m} \implies \boxed{A}$$

13 Problem 13

Pressure increases linearly with depth, so the average pressure from the water on the dam is $\rho g H/2$. Therefore, the total force is

$$F = \overline{P}A = \frac{1}{2}\rho g H^2 L \implies \boxed{C}$$

14 Problem 14

While falling, both masses fall at the same rate. Therefore, the mass within stays at its equilibrium position during the fall. When the box hits the ground, it immediately slows to a stop, whereas the mass within still has a velocity $\sqrt{2gH}$. The maximum compression achieved by the mass within can be determined by conservation of energy:

$$\frac{1}{2}kx^2 = mgH$$

$$x = \sqrt{\frac{2mgH}{k}}$$

For the box to jump up afterward, kx must be greater than mg:

$$kx = \sqrt{2mgHk} > mg$$

$$H > \frac{mg}{2k} \implies \boxed{C}$$

15 Problem 15

Angular momentum is conserved throughout this process, so

$$\frac{d}{dt}L = \frac{d}{dt}(I\omega) = \frac{dI}{dt}\omega + I\frac{d\omega}{dt} = 0$$

$$\frac{d\omega}{dt} = -\frac{dI}{dt}\frac{\omega}{I} = -\frac{1}{2}(0.2 \cdot \pi 1^2)1^2 \cdot \frac{10}{(1/2)2 \cdot 1^2} = -\pi \implies \boxed{D}$$

16 Problem 16

The total momentum, which is conserved, is 0. Therefore, the center of mass is stationary. From inspection, we know that the center of mass is initially within the black curve, so it must always be below the black curve. From these two facts alone, we can eliminate all other choices:

- Choice A satisfies both these requirements. $\implies \boxed{A}$
- Choice B has moments where the center of mass is above of the black curve.
- The initial momentum of Choice C does not even satisfy the problem's requirements.
- Neither does the initial momentum of Choice D.
- In Choice E, the momentum quickly becomes nonzero.

17 Problem 17

In the accelerating reference frame of the center of mass of the firework, the firework bursts into a sphere where each portion of the firework has equal speed. Since fragments with equal height in the sphere will hit the ground at the same time, the number of fragments that hit the ground at a certain time is proportional to the circumference of the circular cross-section of the sphere that intersects with the ground. Therefore, the number of fragments will increase, reach a maximum, then decrease $\implies \boxed{C}$.

18 Problem 18

Adding the springs from top to bottom is annoying to calculate. Instead, we can consider each iteration to be equivalent to taking the existing contraption and hanging it on a new mass, then hanging this new contraption with a spring. That way, it is clear that the increase in length for each iteration of the process is equal to the length of the top spring.

The length of the top spring for the iteration of N springs is:

$$l_0 + \Delta x = l_0 + \frac{Nmg}{k}$$

So the total length of the springs is:

$$\sum_{i=1}^{N} \left(l_0 + \frac{img}{k} \right) = Nl_0 + \frac{N(N-1)}{2}\frac{mg}{k}$$

Setting this total length to be greater than or equal to the height of the room yields a quadratic with solution

$$N = 11 \implies \boxed{D}$$

19 Problem 19

Note that since the center of mass of the table is stationary, all the oscillations described in this problem excite only one normal mode. When point A is pushed down, the motion is equivalent to that of a rod, with two springs of constant $2k$ at each end, and opposite sides of the table alternate going up and down. For a small tilt from the horizontal of angle θ, the torque equation about the center of mass becomes

$$\frac{1}{12}ma^2\ddot{\theta} = -4k\frac{a}{2}\theta \cdot \frac{a}{2}$$

$$\omega_0 = \sqrt{\frac{12k}{m}}$$

When point B is pressed down, the motion is such that the opposite corners alternate going up and down. Note that the moment of inertia across the diagonal of the square, by the perpendicular axis theorem, is equal to the moment of inertia about an axis in the plane of the square that bisects the two opposite sides. However, there are two springs involved instead of four, the lever arm is a factor of $\sqrt{2}$ greater, and the linear elongation in the springs per unit θ is a factor of $\sqrt{2}$ greater, therefore the oscillation frequency is also

$$\omega = \sqrt{\frac{12k}{m}}$$

Therefore $\omega = \omega_0 \implies \boxed{C}$.

20 Problem 20

If there's no friction interaction between the rings, the rotations of each ring does not affect the velocities of the collision. Further, during the collision, friction from the ground does not have time to act on the rings. Therefore, the linear velocities change exactly as if it were an elastic collision.

From this fact, the premise becomes clear. Initially, both rings roll without slipping until the collision, when the ratios of linear to rotational velocity is offset, which allows friction to act on the rings. If friction is sufficient, it may result one of the rings moving in the opposite direction, causing a second or perhaps even more collisions.

Since the masses are equal, the velocities v_1 and v_2 simply swap. After the first collision, the ring A is suddenly rotating too fast ($\omega_1 = 10/R > v/R$) for its linear speed; therefore, friction points forwards to decelerate its angular speed while accelerating its linear speed:

$$ma = +\mu mg \implies a = +\mu g$$
$$mR^2\alpha = -\mu mgR \implies R\alpha = -\mu g$$

Both accelerations stop when $\omega = v/R$, indicating rolling without slipping. Note that due to the nice setup, $a = -R\alpha$, so v and $R\omega$ simply meet at their average, which is 6 m/s. The linear distance travelled during this time is given by the kinematic equation

$$v_f^2 = v_i^2 + 2ad_A$$
$$d_A = \frac{7^2 - 4^2}{2\mu g} = 1.65 \text{ m}$$

A similar analysis on ring B shows that it also ends up at 6 m/s after the collision and friction act on it, but during the same amount of time, ring B travels a distance $d_B = 2.55$ m. Since they both end up with the same velocity after the first collision, the two rings' separation stays constant afterwards. Therefore, the final separation is 0.9 m $\implies \boxed{A}$.

21 Problem 21

Since the cloud is uniform, the amount of dust collected is proportional to the distance x travelled. Suppose the length that can be traversed in time T_0 without dust is L_0. Then $m(x)$ is

$$m(x) = m + \delta m \frac{x}{L_0}$$

By momentum conservation mv is constant. Therefore:

$$v(x) = v_0 \frac{m}{m(x)}$$

The time needed to traverse the length L_0 is

$$\int \frac{dx}{v} = \int_0^{L_0} \frac{dx}{v_0}\left(1 + \frac{\delta m}{m}\frac{x}{L_0}\right) = \frac{L_0}{v_0} + \frac{\delta m}{mv_0 L_0}\frac{L_0^2}{2} = T_0\left(1 + \frac{\delta m}{2m}\right) = 1.01T_0 \implies \boxed{B}$$

Note that calculus is only necessary for the exact solution. For an approximate solution, note that to first order, $v(x)$ is

$$v(x) \approx v_0 \left(1 - \frac{\delta m}{m} \frac{x}{L_0}\right)$$

which is a linear function with average velocity

$$\overline{v} \approx v_0 \left(1 - \frac{\delta m}{2m}\right)$$

After which we get a similar answer as before:

$$T = \frac{L_0}{\overline{v}} \approx T_0 \left(1 + \frac{\delta m}{2m}\right) = 1.01 T_0$$

22 Problem 22

The thrust required to keep the spaceship in that position is

$$F_{\text{net}} = F - \frac{GMm_f}{4R^2} = \frac{dp}{dt} = \frac{dm}{dt} v = \mu v$$

$$F = \frac{GM(m_0 + \mu R/v)}{4R^2} + \mu v = \text{const.} + \mu \left(\frac{GM}{4Rv} + v\right)$$

which by AM-GM is minimized when $v = \sqrt{GM/4R} \implies \boxed{A}$

23 Problem 23

Angular momentum of the spaceship is conserved. Therefore, we know the answer should depend on both ρ_0 and ρ, since as $\rho \to 0$ the time it takes to slow down should go to infinity (since massless dust can't obtain angular momentum and slow down the spaeship), and $\rho_0 \to \infty$ should achieve similar results (since the dust is essentially massless relative to the spaceship). Therefore, in an expression for time, ρ should be in the denominator and ρ_0 should be in the numerator. The only answer choice that satisfies both of these requirements is $\boxed{C}$.

24 Problem 24

There are two velocities involved: the daily rotation of the Earth and the velocity at which Earth revolves around the Sun. The latter is the much greater contributor, and it has a magnitude of

$$\frac{2\pi a}{T} = \frac{2\pi \times 1.5 \times 10^{11} \text{ m}}{365 \times 24 \times 3600 \text{ s}} \approx 3 \times 10^4 \text{ m/s} \implies \boxed{B}$$

25 Problem 25

The angular impulse about the center of mass of the square due to the impulse J has magnitude $Ja/\sqrt{2}$. This causes the square to obtain an angular velocity

$$\omega = \frac{Ja}{I\sqrt{2}}$$

where I is the moment of inertia of the square across the diagonal. Note that by the perpendicular axis theorem, the moment of inertia about a diagonal of a uniform square is equal to the moment of inertia through an axis that bisects two opposite sides. Therefore $I = (2m)a^2/12$ and

$$\omega = \frac{3\sqrt{2}J}{ma}$$

Now, we turn to the rotational dynamics after the impulse. The tensions must provide the centripetal acceleration of each triangle to keep them rotating at a speed ω. The appropriate equation is

$$2T = m\omega^2 r_{\text{cm}}$$

It is well known that the center of mass of a uniform triangle is located at its centroid, which by geometry can be found to be a radial distance $r_{\text{cm}} = \sqrt{2}a/6$ away from the center of rotation. Therefore

$$T = \frac{1}{2}m\frac{18J^2}{m^2a^2}\frac{\sqrt{2}a}{6} = \frac{3\sqrt{2}J^2}{2ma} \implies ???$$